

Relations among pionic decays of spin-1 mesons from an $SU(4) \times U(1)$ emergent symmetry in QCD

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Motivated by recent results by lattice analysis, we assume that the spin-1 mesons of $(\rho, \omega, a_1, \rho', \omega', b_1, f_1, h_1)$ make a representation of **16** of $U(4)$ emergent symmetry in two-flavor QCD when the chiral symmetry is not broken. We study the decay properties of the spin-1 mesons by using a chiral model with an $SU(4) \times U(1)$ hidden local symmetry. We first show that, since the $SU(4)$ symmetry is spontaneously broken together with the chiral symmetry, each coupling of the interaction among one pion and two spin-1 mesons is proportional to the mass difference of the relevant spin-1 mesons similarly to the Goldberger-Treiman relation. In addition, some of one-pion couplings are related with each other by the $SU(4)$ symmetry. We further show that there is a relation among the mass of ρ' meson, the $\rho'\pi\pi$ coupling and the ρ' -photon mixing strength as well as the Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin relation for the ρ meson. From the relations, we give numerical predictions such as ratios of the spin-1 meson decay widths, which are compared with future experiments for testing the existence of the $U(4)$ emergent symmetry.

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I. INTRODUCTION

In Quantum Chromodynamics (QCD) the chiral symmetry is one of the most important symmetry to investigate properties of hadron. In particular, pion is identified as the pseudo Nambu-Goldstone (NG) boson corresponding to the spontaneously symmetry breaking of the chiral symmetry, which means that dynamics of pion is described by the low energy theorem. On the other hand, there are the spin-1 mesons at about 1 GeV mass, as shown in FIG. 1. It is a variable problem to describe the spin-1 mesons and pion in a discussion of symmetries and their breaking.

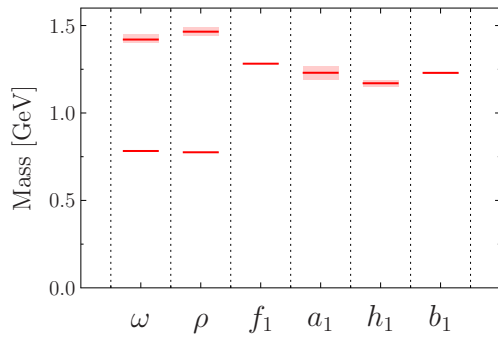


FIG. 1: Spectra of spin-1 mesons [1].

Recently, the existence of an $SU(4)$ symmetry in the spin-1 meson sector is suggested by Refs. [2–4] via the

lattice QCD calculation. This is also shown in unbroken limit of the chiral symmetry in two-flavor case in Ref. [5]. In the references, the symmetry is called as an emergent symmetry in QCD. The $SU(4)$ symmetry includes the chiral symmetry $SU(2)_L \times SU(2)_R \times U(1)_A$, which is corresponding to the rotation of the quark field ψ written as $\psi^T = (u_R, d_R, u_L, d_L)$. This means that the mesons denoted by $(\rho, a_1, \rho', \omega', b_1, f_1, h_1)$ are members belonging to a multiplet of the $SU(4)$ symmetry, and that the mass differences of members are caused by the spontaneous chiral symmetry breaking. This indicates that there exist extended Goldberger-Treiman (GT) relations between the mass differences of spin-1 mesons and their couplings to pions which are Nambu-Goldstone bosons associated with the chiral symmetry breaking.

In this work, we construct an effective Lagrangian with an $SU(4) \times U(1)$ hidden local symmetry (HLS) [6–11], which includes the spin-1 mesons, $(\omega, \rho, a_1, \rho', \omega', b_1, f_1, h_1)$, as gauge fields of the HLS. The symmetry of the Lagrangian is $[SU(2)_R \times SU(2)_L \times U(1)_A]_{\text{chiral}} \times [SU(4) \times U(1)]_{\text{HLS}}$, which is broken to $SU(2)_{\text{isospin}}$ symmetry by the chiral condensate. Then, we show extended GT relations, by which each coupling of the interaction among one pion and two spin-1 mesons is proportional to the mass difference of the relevant spin-1 mesons. In addition, we can derive relations among the coupling of the spin-1 mesons to one pion thanks to the existence of the $SU(4)$ symmetry. Furthermore, we show that there is a relation among the mass of ρ' meson, the $\rho'\pi\pi$ coupling and the ρ' -photon mixing strength as well as the Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin relations for the ρ meson. These relations give us predictions for one-pion decay widths of spin-1 mesons and the electromagnetic form factor of the pion, which can be verified by future experiments.

In this paper, we conduct the analyses: In Sec. II, we

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construct a Lagrangian with the $SU(4) \times U(1)$ HLS to introduce the spin-1 mesons. In Sec. III, we give eigenstates and masses of the spin-1 mesons. In Sec. IV, we obtain extended GT relations. We study one-pion decays of the spin-1 mesons in section V and extended KSRF relations in section VI. In Sec. VII, we make a numerical analysis to determine the parameters and give a prediction on the electromagnetic form factor of pion. The summary and discussions are given in Sec. VIII.

II. CONSTRUCTION

We construct a chiral Lagrangian with the $SU(4) \times U(1)$ hidden local symmetry. The Lagrangian has the chiral symmetry $^{#1} SU(2)_R \times SU(2)_L \times U(1)_A$ and the gauge symmetry $G_{\text{local}} = [SU(4) \times U(1)]_{\text{HLS}}$. Here spin-1 mesons are introduced as the gauge fields of G_{local} , which are identified as $(\omega, \rho, a_1, \rho', \omega', b_1, f_1, h_1)$ mesons. The spontaneously symmetry breaking is represented as $G_{\text{global}} \times G_{\text{local}} (= [SU(2)_R \times SU(2)_L \times U(1)_A]_{\text{chiral}} \times [SU(4) \times U(1)]_{\text{HLS}}) \rightarrow H (= SU(2)_V)$, where the NG-bosons identified as the pions and the eta meson emerge.

The NG bosons associated with the coset-space G_{global}/H are introduced through the 2 by 2 unitary matrix field U as

$$U = \exp \left(i \frac{\eta}{f_\eta} + i \sum_{a=1}^3 \frac{\pi^a \sigma^a}{f_\pi} \right), \quad (\text{II.1})$$

where η and π^a ($a = 1, 2, 3$) are the eta meson and the pion fields, and σ^a is the Pauli matrix. For introducing the $SU(4) \times U(1)$ HLS, we embed this U into 4 by 4 matrix field \mathcal{U} as

$$\mathcal{U} = \begin{pmatrix} 0 & U^\dagger \\ U & 0 \end{pmatrix} \quad (\text{II.2})$$

which transforms under the chiral symmetry $SU(2)_L \times SU(2)_R \times U(1)_A$ as

$$\mathcal{U} \rightarrow \mathcal{G} \cdot \mathcal{U} \cdot \mathcal{G}^\dagger, \quad (\text{II.3})$$

where \mathcal{G} is an element of $G_{\text{global}} = SU(2)_L \times SU(2)_R \times U(1)_A$ written as

$$\mathcal{G} = \begin{pmatrix} g_R & 0 \\ 0 & g_L \end{pmatrix} \begin{pmatrix} g_A & 0 \\ 0 & g_A^\dagger \end{pmatrix} \quad (\text{II.4})$$

by using $g_{L,R} \in SU(2)_{L,R}$ and $g_A \in U(1)_A$. The genera-

tors of the chiral symmetry G_{global} are

$$\begin{aligned} T_{\text{global}}^A &= \left\{ \frac{S^a + X_{(3)}^a}{\sqrt{2}}, \frac{S^a - X_{(3)}^a}{\sqrt{2}}, X_{(3)}^0 \right\} \\ &= \left\{ T_R^a, T_L^a, X_{(3)}^0 \right\}, \end{aligned} \quad (\text{II.5})$$

whose explicit form are given in Appendix A. After the spontaneous symmetry breaking of G_{global} , the generators expressed by S are corresponding to the unbroken ones, while X are broken generators. This implies that transformations generated by S belong to H .

Let us decompose \mathcal{U} as

$$\mathcal{U} = \Xi^\dagger(x) \cdot \Xi_m^\dagger(x) \cdot \bar{\Sigma} \cdot \Xi_m(x) \cdot \Xi(x) \quad (\text{II.6})$$

by using

$$\bar{\Sigma} \equiv \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}. \quad (\text{II.7})$$

These fields transform under $G_{\text{global}} \times G_{\text{local}} \times H_{\text{extra}}$ as

$$\begin{aligned} \Xi(x) &\rightarrow \tilde{\mathcal{G}}(x) \cdot \Xi(x) \cdot \mathcal{G}^\dagger, \\ \Xi_m(x) &\rightarrow \tilde{h}(x) \cdot \Xi_m(x) \cdot \tilde{\mathcal{G}}^\dagger(x) \end{aligned} \quad (\text{II.8})$$

where $\mathcal{G} \in G_{\text{global}}$, $\tilde{\mathcal{G}}(x) \in G_{\text{local}}$, and $\tilde{h}(x)$ is an element of an $H_{\text{extra}} (= U(2))$ extra local symmetry, whose generator is written as

$$T_{\text{extra}}^A = \{S^a, S^0\}. \quad (\text{II.9})$$

From the above transformation properties, the covariant derivatives are expressed as

$$\begin{aligned} D_\mu \Xi(x) &= \partial_\mu \Xi(x) - i V_\mu \Xi(x) + i \Xi(x) \mathcal{V}_\mu \\ D_\mu \Xi_m(x) &= \partial_\mu \Xi_m(x) - i \tilde{V}_\mu \Xi_m(x) + i \Xi_m(x) V_\mu \end{aligned} \quad (\text{II.10})$$

where V_μ is the HLS gauge field, $V_\mu = V_\mu^A T^A$, for G_{local} , \mathcal{V}_μ is the external gauge field written by

$$\mathcal{V}_\mu = \mathcal{R}_\mu^a \cdot T_R^a + \mathcal{L}_\mu^a \cdot T_L^a + \sqrt{2} \mathcal{V}_\mu^0 S^0 + \sqrt{2} \mathcal{A}_\mu^0 X_{\perp(3)}^0, \quad (\text{II.11})$$

and \tilde{V}_μ is the gauge field for H_{extra} . We note that we do not introduce the kinetic term for this \tilde{V}_μ , so that it is not a dynamical field in the present analysis. The field strength of the HLS gauge field is written as

$$V_{\mu\nu} \equiv \partial_\mu V_\nu - \partial_\nu V_\mu - i [V_\mu, V_\nu]. \quad (\text{II.12})$$

For constructing the Lagrangian, it is convenient to introduce the following covariantized Maurer-Cartan 1-forms:

$$\begin{aligned} \hat{\alpha}_\mu(x) &\equiv \frac{1}{i} \Xi_m(x) \cdot (D_\mu \Xi(x) \cdot \Xi^\dagger(x)) \cdot \Xi_m^\dagger(x), \\ \hat{\alpha}_\mu^{(m)}(x) &\equiv \frac{1}{i} D_\mu \Xi_m(x) \cdot \Xi_m^\dagger(x), \end{aligned} \quad (\text{II.13})$$

^{#1} Explicit breaking of $U(1)_A$ by anomaly is added later together with the explicit chiral symmetry breaking from the current quark masses of up and down quarks.

Table I: Transformation properties of the Maurer-Cartan 1-forms $\hat{\alpha}_\mu$ and $\hat{\alpha}_\mu^{(m)}$ under $G_{\text{global}} \times G_{\text{local}}$, \mathcal{P} , and \mathcal{C} transformations.

$G_{\text{global}} \times G_{\text{local}}$	$\hat{\alpha}_\mu(x) \rightarrow \tilde{h}(x) \cdot \hat{\alpha}_\mu(x) \cdot \tilde{h}^\dagger(x)$
\mathcal{P} trans.	$\hat{\alpha}_\mu(x) \xrightarrow{\mathcal{P}} \bar{\Sigma} \cdot \hat{\alpha}_\mu(x) \cdot \bar{\Sigma}$
\mathcal{C} trans.	$\hat{\alpha}_\mu(x) \xrightarrow{\mathcal{C}} -\bar{\Sigma} \cdot (\hat{\alpha}_\mu(x))^* \cdot \bar{\Sigma}$
$G_{\text{global}} \times G_{\text{local}}$	$\hat{\alpha}_\mu^{(m)}(x) \rightarrow \tilde{h}(x) \cdot \hat{\alpha}_\mu^{(m)}(x) \cdot \tilde{h}^\dagger(x)$
\mathcal{P} trans.	$\hat{\alpha}_\mu^{(m)}(x) \xrightarrow{\mathcal{P}} \bar{\Sigma} \cdot \hat{\alpha}_\mu^{(m)}(x) \cdot \bar{\Sigma}$
\mathcal{C} trans.	$\hat{\alpha}_\mu^{(m)}(x) \xrightarrow{\mathcal{C}} -\bar{\Sigma} \cdot (\hat{\alpha}_\mu^{(m)}(x))^* \cdot \bar{\Sigma}$

which transform under $G_{\text{global}} \times G_{\text{local}} \times H_{\text{extra}}$, the parity \mathcal{P} , and the charge conjugation \mathcal{C} as in Table I.

Now, the 1-form $\hat{\alpha}_\mu$ is classified as

$$\begin{aligned}
\hat{\alpha}_{\mu\parallel}(x) &\equiv 2\text{Tr} [\hat{\alpha}_\mu \cdot S^a] S^a, \\
\hat{\alpha}_{\mu\perp(1)}(x) &\equiv 2\text{Tr} [\hat{\alpha}_\mu \cdot X_{(1)}^a] X_{(1)}^a, \\
\hat{\alpha}_{\mu\perp(2)}(x) &\equiv 2\text{Tr} [\hat{\alpha}_\mu \cdot X_{(2)}^a] X_{(2)}^a, \\
\hat{\alpha}_{\mu\perp(3)}(x) &\equiv 2\text{Tr} [\hat{\alpha}_\mu \cdot X_{(3)}^a] X_{(3)}^a, \\
\hat{\alpha}_{\mu\parallel}^{(I=0)}(x) &\equiv 2\text{Tr} [\hat{\alpha}_\mu \cdot S^0] S^0, \\
\hat{\alpha}_{\mu\perp(1)}^{(I=0)}(x) &\equiv 2\text{Tr} [\hat{\alpha}_\mu \cdot X_{(1)}^0] X_{(1)}^0, \\
\hat{\alpha}_{\mu\perp(2)}^{(I=0)}(x) &\equiv 2\text{Tr} [\hat{\alpha}_\mu \cdot X_{(2)}^0] X_{(2)}^0, \\
\hat{\alpha}_{\mu\perp(3)}^{(I=0)}(x) &\equiv 2\text{Tr} [\hat{\alpha}_\mu \cdot X_{(3)}^0] X_{(3)}^0,
\end{aligned} \tag{II.14}$$

and similarly for $\alpha_\mu^{(m)}$. Note that the \tilde{V}_μ gauge field for H_{extra} is included only in $\hat{\alpha}_{\mu\parallel}^{(m)}$ and $\hat{\alpha}_{\mu\parallel}^{(m)(I=0)}$. Furthermore, from the definitions (II.13) of $\hat{\alpha}_\mu$ and $\hat{\alpha}_\mu^{(m)}$, we find that the sum

$$\begin{aligned}
\hat{\alpha}_\mu(x) + \hat{\alpha}_\mu^{(m)}(x) &= \frac{1}{i} D_\mu [\Xi_m(x) \cdot \Xi(x)] \cdot [\Xi_m(x) \cdot \Xi(x)]^\dagger \\
&\in \{X_{(3)}^A, S^A\}
\end{aligned} \tag{II.15}$$

is expanded in terms of $X_{(3)}^A$ and $S^A (= T_{\text{extra}}^A)$ only because the Maurer-Cartan 1-forms of $\Xi_m(x) \cdot \Xi(x)$ are constructed by the broken generators corresponding to G_{global}/H and H_{extra} . This relation yield

$$\begin{aligned}
\hat{\alpha}_{\mu\perp(1)}(x) + \hat{\alpha}_{\mu\perp(1)}^{(m)}(x) &= 0, \\
\hat{\alpha}_{\mu\perp(2)}(x) + \hat{\alpha}_{\mu\perp(2)}^{(m)}(x) &= 0, \\
\hat{\alpha}_{\mu\perp(1)}^{(I=0)}(x) + \hat{\alpha}_{\mu\perp(1)}^{(m)(I=0)}(x) &= 0, \\
\hat{\alpha}_{\mu\perp(2)}^{(I=0)}(x) + \hat{\alpha}_{\mu\perp(2)}^{(m)(I=0)}(x) &= 0.
\end{aligned} \tag{II.16}$$

To take account of the effect of current quark masses, we introduce an external source χ which transforms under the chiral symmetry G_{global} as

$$\chi \rightarrow \mathcal{G} \cdot \chi \cdot \mathcal{G}^\dagger. \tag{II.17}$$

We assume that its expectation value is given as

$$\langle \chi \rangle = m_\pi^2 \bar{\Sigma} = \begin{pmatrix} 0 & m_\pi^2 1_2 \\ m_\pi^2 1_2 & 0 \end{pmatrix}. \tag{II.18}$$

We redefine the external source as

$$\hat{\chi} \equiv \Xi_m(x) \cdot \Xi(x) \cdot \chi \cdot \Xi^\dagger(x) \cdot \Xi_m^\dagger(x) \tag{II.19}$$

such that it transforms under $G_{\text{global}} \times G_{\text{local}}$, \mathcal{P} , and \mathcal{C} as

$$\begin{aligned}
\hat{\chi} &\rightarrow \tilde{h}(x) \cdot \hat{\chi} \cdot \tilde{h}^\dagger(x), \\
\hat{\chi} &\xrightarrow{\mathcal{P}} \bar{\Sigma} \cdot \hat{\chi} \cdot \bar{\Sigma}, \\
\hat{\chi} &\xrightarrow{\mathcal{C}} \bar{\Sigma} \cdot (\hat{\chi})^* \cdot \bar{\Sigma},
\end{aligned} \tag{II.20}$$

respectively.

By using a standard order counting manner for the fields:

$$V_\mu \sim \tilde{V}_\mu \sim \mathcal{O}(p), \quad \alpha_\mu \sim \mathcal{O}(p), \quad \hat{\chi} \sim \mathcal{O}(p^2), \tag{II.21}$$

possible operators invariant under $G_{\text{global}} \times G_{\text{local}} (\times H_{\text{extra}})$ as well as \mathcal{P} and \mathcal{C} at $\mathcal{O}(p^2)$ which do not include $\hat{\alpha}_{\parallel}^{(m)}(x)$ are $\hat{\alpha}_{\parallel}^{(m)(I=0)}(x)$ are written as

$$\begin{aligned}
\mathcal{L}_1 &= F^2 \text{Tr} \left[\left(\hat{\alpha}_{\mu\perp(3)}(x) + \hat{\alpha}_{\mu\perp(3)}^{(m)}(x) \right)^2 \right], \\
\mathcal{L}_2 &= F^2 \text{Tr} \left[\left(\hat{\alpha}_{\mu\parallel}(x) \right)^2 \right], \\
\mathcal{L}_3 &= F^2 \text{Tr} \left[\left(\hat{\alpha}_{\mu\perp(1)}^{(m)}(x) \right)^2 \right], \\
\mathcal{L}_4 &= F^2 \text{Tr} \left[\left(\hat{\alpha}_{\mu\perp(2)}^{(m)}(x) \right)^2 \right], \\
\mathcal{L}_5 &= F^2 \text{Tr} \left[\left(\hat{\alpha}_{\mu\perp(3)}^{(m)}(x) \right)^2 \right], \\
\mathcal{L}_6 &= 2F^2 \text{Tr} \left[\hat{\alpha}_{\mu\parallel}^{(m)}(x) \cdot \hat{\alpha}_{\perp(1)}^{(m)\mu}(x) \cdot \bar{\Sigma} \right], \\
\mathcal{L}_7 &= -2F^2 \text{Tr} \left[\left(\hat{\alpha}_{\mu\perp(3)}(x) + \hat{\alpha}_{\mu\perp(3)}^{(m)}(x) \right) \cdot \hat{\alpha}_{\perp(3)}^{(m)\mu}(x) \right], \\
\mathcal{L}_8 &= F^2 \text{Tr} \left[\left(\hat{\alpha}_{\mu\perp(3)}^{(I=0)}(x) + \hat{\alpha}_{\mu\perp(3)}^{(m)(I=0)}(x) \right)^2 \right], \\
\mathcal{L}_9 &= F^2 \text{Tr} \left[\left(\hat{\alpha}_{\mu\parallel}^{(I=0)}(x) \right)^2 \right], \\
\mathcal{L}_{10} &= F^2 \text{Tr} \left[\left(\hat{\alpha}_{\mu\perp(1)}^{(m)(I=0)}(x) \right)^2 \right], \\
\mathcal{L}_{11} &= F^2 \text{Tr} \left[\left(\hat{\alpha}_{\mu\perp(2)}^{(m)(I=0)}(x) \right)^2 \right], \\
\mathcal{L}_{12} &= F^2 \text{Tr} \left[\left(\hat{\alpha}_{\mu\perp(3)}^{(m)(I=0)}(x) \right)^2 \right], \\
\mathcal{L}_{13} &= 2F^2 \text{Tr} \left[\hat{\alpha}_{\mu\parallel}^{(I=0)}(x) \cdot \hat{\alpha}_{\perp(1)}^{(m)(I=0)\mu}(x) \cdot \bar{\Sigma} \right], \\
\mathcal{L}_{14} &= -2F^2 \text{Tr} \left[\left(\hat{\alpha}_{\mu\perp(3)}^{(I=0)}(x) + \hat{\alpha}_{\mu\perp(3)}^{(m)(I=0)}(x) \right) \cdot \hat{\alpha}_{\perp(3)}^{(m)(I=0)\mu}(x) \right], \\
\mathcal{L}_\chi &= F^2 \text{Tr} [\hat{\chi} \cdot \bar{\Sigma}],
\end{aligned} \tag{II.22}$$

where F is a constant with dimension one. Note that this F is not the pion decay constant, which will be determined later. In addition, there are eight operators including $\hat{\alpha}_{\parallel}^{(m)}(x)$ or $\hat{\alpha}_{\parallel}^{(m)(I=0)}(x)$:

$$\begin{aligned}
\mathcal{L}'_1 &= F^2 \text{Tr} \left[\left(\hat{\alpha}_{\parallel}^{(m)}(x) \right)^2 \right], \\
\mathcal{L}'_2 &= 2F^2 \text{Tr} \left[\hat{\alpha}_{\parallel}^{(m)}(x) \cdot \hat{\alpha}_{\parallel}^{(m)}(x) \right], \\
\mathcal{L}'_3 &= 2F^2 \text{Tr} \left[\hat{\alpha}_{\parallel}^{(m)}(x) \cdot \hat{\alpha}_{\perp(1)}^{\mu}(x) \cdot \bar{\Sigma} \right], \\
\mathcal{L}'_4 &= 2F^2 \text{Tr} \left[\hat{\alpha}_{\parallel}^{(m)}(x) \cdot \hat{\alpha}_{\perp(1)}^{(m)\mu}(x) \cdot \bar{\Sigma} \right], \\
\mathcal{L}'_5 &= F^2 \text{Tr} \left[\left(\hat{\alpha}_{\parallel}^{(m)(I=0)}(x) \right)^2 \right], \\
\mathcal{L}'_6 &= 2F^2 \text{Tr} \left[\hat{\alpha}_{\parallel}^{(m)(I=0)}(x) \cdot \hat{\alpha}_{\parallel}^{(I=0)\mu}(x) \right], \\
\mathcal{L}'_7 &= 2F^2 \text{Tr} \left[\hat{\alpha}_{\parallel}^{(m)(I=0)}(x) \cdot \hat{\alpha}_{\perp(1)}^{(I=0)\mu}(x) \cdot \bar{\Sigma} \right], \\
\mathcal{L}'_8 &= 2F^2 \text{Tr} \left[\hat{\alpha}_{\parallel}^{(m)(I=0)}(x) \cdot \hat{\alpha}_{\perp(1)}^{(m)(I=0)\mu}(x) \cdot \bar{\Sigma} \right]. \quad (\text{II.23})
\end{aligned}$$

Although the term given by

$$F^2 \text{Tr} [\hat{\chi}] \quad (\text{II.24})$$

is also an allowed operator, it contributes only to the vacuum energy. The Lagrangian with the $\text{SU}(4) \times \text{U}(1)$ HLS at $\mathcal{O}(p^2)$ is written as

$$\begin{aligned}
\tilde{\mathcal{L}}_{[\text{SU}(4) \times \text{U}(1)]_{\text{HLS}}}^{\mathcal{O}(p^2)} &= \sum_{n=1}^{14} a_{(n)} \mathcal{L}_n + \sum_{n=1}^8 b_{(n)} \mathcal{L}'_n \\
&\quad - \frac{1}{2g^2} \text{Tr} [V_{\mu\nu} V^{\mu\nu}] \\
&\quad - \left(\frac{1}{g_B^2} - \frac{1}{g^2} \right) \text{Tr} [V_{\mu\nu}] \text{Tr} [V^{\mu\nu}] \\
&\quad + a_{\chi} \mathcal{L}_{\chi} \quad (\text{II.25})
\end{aligned}$$

with arbitrary real coefficients $a_{(n)}$, $b_{(n)}$, and a_{χ} . g and g_B are the gauge coupling corresponding to $\text{SU}(4)_{\text{HLS}}$ and $\text{U}(1)_{\text{HLS}}$, respectively^{#2}.

In the above Lagrangian (II.25), $\Xi(x)$ and $\Xi_m(x)$ are parametrized as

$$\begin{aligned}
\Xi(x) &\equiv e^{-ip/F_p} \cdot e^{is/F_s} \cdot e^{i\pi/F_{\pi}} = \Xi(p)^{\dagger} \cdot \Xi(s) \cdot \Xi(\pi), \\
\Xi_m(x) &\equiv e^{-i\tilde{s}/F_{\tilde{s}}} \cdot e^{ip/F_p} = \Xi^{\dagger}(\tilde{s}) \cdot \Xi(p) \quad (\text{II.26})
\end{aligned}$$

where the p , s , \tilde{s} , and π are

$$\begin{aligned}
p(x) &= p_{(1)}^A(x) \cdot X_{(1)}^A + p_{(2)}^A(x) \cdot X_{(2)}^A + p_{(3)}^A(x) \cdot X_{(3)}^A, \\
s(x) &= s_s^A(x) \cdot S^A, \quad \tilde{s}(x) = \tilde{s}_s^A(x) \cdot S^A, \\
\pi(x) &= \pi^A(x) \cdot X_{(3)}^A, \quad (\text{II.27})
\end{aligned}$$

respectively. $\pi(x)$ is the NG-boson field corresponding to the breaking of the chiral symmetry, which is identified with the pion. F_p , F_s , F_{π} , and $F_{\tilde{s}}$ are constants with one mass-dimension, in particular F_{π} is the pion decay constant. $p(x)$, $s(x)$, and $\tilde{s}(x)$ are also the NG-boson fields which are eaten by the gauge fields.

Since \tilde{V}_{μ} included in $\hat{\alpha}_{\parallel}^{(m)}$ is not a dynamical field, we fix $\tilde{s}(x) = 0$ and integrate out the gauge field. Then \mathcal{L}'_n become the terms given in Eqs. (II.22), and the Lagrangian is written by

$$\mathcal{L}_{[\text{SU}(4) \times \text{U}(1)]_{\text{HLS}}}^{\mathcal{O}(p^2)} = \mathcal{L}_V + \mathcal{L}_k + a_{\chi} \mathcal{L}_{\chi}, \quad (\text{II.28})$$

where

$$\begin{aligned}
\mathcal{L}_V &= \sum_{n=1}^{14} \bar{a}_{(n)} \mathcal{L}_n \\
\mathcal{L}_k &= -\frac{1}{2g^2} \text{Tr} [V_{\mu\nu} V^{\mu\nu}] \\
&\quad - \left(\frac{1}{g_B^2} - \frac{1}{g^2} \right) \text{Tr} [V_{\mu\nu}] \text{Tr} [V^{\mu\nu}], \quad (\text{II.29})
\end{aligned}$$

with the coefficients $\bar{a}_{(n)}$ being certain linear combinations of $a_{(n)}$ and $b_{(n)}$.

To analyze the dynamics of the spin-1 mesons together with the pion in the model, in the following analysis, we take the unitary gauge

$$p = s = 0 \quad (\text{II.30})$$

as well as $\tilde{s} = 0$. As shown in Appendix B, the expanded form of the Maurer-Cartan 1-forms are written by using π and V_{μ} .

III. EIGENSTATES AND MASSES

In this section, we obtain the mass eigenstates of the spin-1 mesons and their masses.

By using the generators of $\text{SU}(4) \times \text{U}(1)$ listed in Appendix A, the HLS gauge field is decomposed as

$$\begin{aligned}
V_{\mu} &= V_{\mu\parallel} + V_{\mu\perp(1)} + V_{\mu\perp(2)} + V_{\mu\perp(3)} \\
&\quad + V_{\mu\parallel}^{(I=0)} + V_{\mu\perp(1)}^{(I=0)} + V_{\mu\perp(2)}^{(I=0)} + V_{\mu\perp(3)}^{(I=0)}, \quad (\text{III.1})
\end{aligned}$$

where

$$\begin{aligned}
V_{\mu\parallel} &\equiv 2\text{Tr} [V_{\mu} \cdot S^a] S^a \equiv V_{\mu\parallel}^a \cdot S^a, \\
V_{\mu\parallel}^{(I=0)} &\equiv 2\text{Tr} [V_{\mu} \cdot S^0] S^0 \equiv V_{\mu\parallel}^0 \cdot S^0, \\
V_{\mu\perp(i)} &\equiv 2\text{Tr} [V_{\mu} \cdot X_{(i)}^a] X_{(i)}^a \equiv V_{\mu\perp(i)}^a \cdot X_{(i)}^a, \\
V_{\mu\perp(i)}^{(I=0)} &\equiv 2\text{Tr} [V_{\mu} \cdot X_{(i)}^0] X_{(i)}^0 \equiv V_{\mu\perp(i)}^0 \cdot X_{(i)}^0. \quad (\text{III.2})
\end{aligned}$$

They are classified as $(\rho, \omega, a_1, \rho', \omega', b_1, f_1, h_1)$ by the properties of transformation under \mathcal{P} and \mathcal{C} . Because

^{#2} The values of the couplings are scaled by $\sqrt{2}$ comparing with usual way as defined in Ref. [11].

the fields satisfy

$$\begin{aligned}
V_\mu &\xrightarrow{\mathcal{P}} \bar{\Sigma} \cdot V^\mu \cdot \bar{\Sigma} \\
&= V_\parallel^\mu + V_{\perp(1)}^\mu - V_{\perp(2)}^\mu - V_{\perp(3)}^\mu \\
&\quad + V_\parallel^{\mu(I=0)} + V_{\perp(1)}^{\mu(I=0)} - V_{\perp(2)}^{\mu(I=0)} - V_{\perp(3)}^{\mu(I=0)} , \\
V_\mu &\xrightarrow{\mathcal{C}} -\bar{\Sigma} \cdot (V_\mu)^* \cdot \bar{\Sigma} \\
&= - (V_\parallel)^* - (V_{\mu\perp(1)})^* - (V_{\mu\perp(2)})^* + (V_{\mu\perp(3)})^* \\
&\quad - (V_\parallel^{(I=0)})^* - (V_{\mu\perp(1)}^{(I=0)})^* - (V_{\mu\perp(2)}^{(I=0)})^* + (V_{\mu\perp(3)}^{(I=0)})^* , \\
&\hspace{15em} \text{(III.3)}
\end{aligned}$$

$$\begin{aligned}
(V_\parallel, V_{\mu\perp(1)}) &\Rightarrow \rho, \rho' , \\
(V_\parallel^{(I=0)}, V_{\mu\perp(1)}^{(I=0)}) &\Rightarrow \omega, \omega' , \\
V_{\mu\perp(2)} &\Rightarrow b_1 , \quad V_{\mu\perp(2)}^{(I=0)} \Rightarrow h_1 , \\
V_{\mu\perp(3)} &\Rightarrow a_1 , \quad V_{\mu\perp(3)}^{(I=0)} \Rightarrow f_1 . \\
&\hspace{15em} \text{(III.4)}
\end{aligned}$$

From $\mathcal{L}_V + a_\chi \mathcal{L}_\chi$, the quadratic terms with respect to the fields are given as

$$\begin{aligned}
\mathcal{L}_V + a_\chi \mathcal{L}_\chi &= \frac{1}{2} \frac{F^2}{F_\pi^2} \left(\bar{a}_{(1)} - \frac{\bar{a}_{(7)}^2}{\bar{a}_{(5)}} \right) (\partial_\mu \pi^a)^2 + \frac{1}{2} \left(a_\chi \frac{F^2}{F_\pi^2} \right) m_\pi^2 (\pi^a)^2 + \frac{1}{2} \frac{F^2}{F_\pi^2} \left(\bar{a}_{(8)} - \frac{\bar{a}_{(14)}^2}{\bar{a}_{(12)}} \right) (\partial_\mu \eta)^2 + \frac{1}{2} \left(a_\chi \frac{F^2}{F_\pi^2} \right) m_\pi^2 (\eta)^2 \\
&\quad + \sqrt{2} \frac{F^2}{F_\pi} \left(\bar{a}_{(1)} - \frac{\bar{a}_{(7)}^2}{\bar{a}_{(5)}} \right) (\mathcal{A}_\mu^a \partial^\mu \pi^a) + \sqrt{2} \frac{F^2}{F_\pi} \left(\bar{a}_{(8)} - \frac{\bar{a}_{(14)}^2}{\bar{a}_{(12)}} \right) (\mathcal{A}_\mu^0 \partial^\mu \eta) \\
&\quad + \frac{1}{2} (\bar{a}_{(4)} F^2 g^2) \frac{1}{g^2} (V_{\mu\perp(2)}^a)^2 + \frac{1}{2} (\bar{a}_{(11)} F^2 g^2) \frac{1}{g^2} (V_{\mu\perp(2)}^0)^2 \\
&\quad + \frac{1}{2} (\bar{a}_{(5)} F^2 g^2) \frac{1}{g^2} \left(V_{\mu\perp(3)}^a - \frac{\bar{a}_{(7)}}{\bar{a}_{(5)}} \frac{1}{F_\pi} \partial_\mu \pi^a \right)^2 + \frac{1}{2} (\bar{a}_{(12)} F^2 g^2) \frac{1}{g^2} \left(V_{\mu\perp(3)}^0 - \frac{\bar{a}_{(14)}}{\bar{a}_{(12)}} \frac{1}{F_\pi} \partial_\mu \eta \right)^2 \\
&\quad + g^2 F^2 \text{Tr} \left[\left(\frac{1}{g} V_\parallel, \frac{1}{g} \bar{\Sigma} \cdot V_{\mu\perp(1)} \right) \begin{pmatrix} \bar{a}_{(2)} & -\bar{a}_{(6)} \\ -\bar{a}_{(6)} & \bar{a}_{(3)} \end{pmatrix} \begin{pmatrix} \frac{1}{g} V_\parallel \\ \frac{1}{g} \bar{\Sigma} \cdot V_{\mu\perp(1)} \end{pmatrix} \right] \\
&\quad + g^2 F^2 \text{Tr} \left[\left(\frac{1}{g_B} V_\parallel^{(I=0)}, \frac{1}{g} \bar{\Sigma} \cdot V_{\mu\perp(1)}^{(I=0)} \right) \begin{pmatrix} \frac{g_B^2}{g^2} \bar{a}_{(9)} & -\frac{g_B}{g} \bar{a}_{(13)} \\ -\frac{g_B}{g} \bar{a}_{(13)} & \bar{a}_{(10)} \end{pmatrix} \begin{pmatrix} \frac{1}{g_B} V_\parallel^{(I=0)} \\ \frac{1}{g} \bar{\Sigma} \cdot V_{\mu\perp(1)}^{(I=0)} \end{pmatrix} \right] + \dots \\
&\hspace{15em} \text{(III.5)}
\end{aligned}$$

where the axial external gauge field is defined as $\mathcal{A}_\mu^a = \frac{1}{2} (\mathcal{R}_\mu^a - \mathcal{L}_\mu^a)$. Note that the field η defined by $\eta \equiv 2\text{Tr} [\pi \cdot X_{(3)}^0]$ is the linear combination of the lowest eta meson and $\eta'(958)$. To normalize the kinetic terms of π^a and η , we set

$$F^2 \left(\bar{a}_{(1)} - \frac{\bar{a}_{(7)}^2}{\bar{a}_{(5)}} \right) = F_\pi^2 , \quad F^2 \left(\bar{a}_{(8)} - \frac{\bar{a}_{(14)}^2}{\bar{a}_{(12)}} \right) = F_\pi^2 , \quad \text{(III.6)}$$

together with $a_\chi F^2 = F_\pi^2$ which makes the pion mass be m_π . The second line in Eq. (III.5) implies that the physical pion decay constant is defined as

$$f_\pi \equiv \sqrt{2} F_\pi , \quad \text{(III.7)}$$

whose value is given in Table III. Furthermore, the mass eigenstates of a_1 and f_1 are defined by

$$(a_1)_\mu \equiv \frac{1}{g} \left(V_{\mu\perp(3)}^a - \frac{r_{a_1}}{f_\pi} \partial_\mu \pi^a \right) X_{(3)}^a , \quad (f_1)_\mu \equiv \frac{1}{g} \left(V_{\mu\perp(3)}^0 - \frac{r_{f_1}}{f_\pi} \partial_\mu \eta \right) X_{(3)}^0 , \quad \text{(III.8)}$$

where

$$r_{a_1} \equiv \sqrt{2} \frac{\bar{a}_{(7)}}{\bar{a}_{(5)}} , \quad r_{f_1} \equiv \sqrt{2} \frac{\bar{a}_{(14)}}{\bar{a}_{(12)}} . \quad \text{(III.9)}$$

r_{a_1} (r_{f_1}) expresses the mixing rate between the a_1 (f_1) meson and the pion π (η). Their masses are obtained as

$$m_{a_1}^2 = \bar{a}_{(5)} g^2 F^2 , \quad m_{f_1}^2 = \bar{a}_{(12)} g^2 F^2 . \quad \text{(III.10)}$$

The mixing of Eq. (III.8) implies that the currents corresponding to the generators $X_{(3)}^a$ and $X_{(3)}^0$ of the SU(4) HLS are coupled to the axial current of the chiral symmetry with the factors of r_{a_1} and r_{f_1} , respectively.

The physical states and masses for b_1 and h_1 are defined as

$$(b_1)_\mu \equiv \frac{1}{g} V_{\mu\perp(2)} , \quad m_{b_1}^2 = \bar{a}_{(4)} g^2 F^2 ,$$

$$(h_1)_\mu \equiv \frac{1}{g} V_{\mu\perp(2)}^{(I=0)} \quad m_{h_1}^2 = \bar{a}_{(11)} g^2 F^2, \quad (\text{III.11})$$

respectively.

By diagonalizing the mass matrices from Eq. (III.5), the eigenstates for the vector mesons are expressed as

$$\begin{aligned} \rho_\mu &= \rho_\mu^a S^a \equiv \frac{1}{g} (\cos \theta_\rho V_{\mu\parallel} - \sin \theta_\rho \bar{\Sigma} \cdot V_{\mu\perp(1)}) , \\ (\rho')_\mu &= (\rho')_\mu^a S^a \equiv \frac{1}{g} (\cos \theta_\rho \bar{\Sigma} \cdot V_{\mu\perp(1)} + \sin \theta_\rho V_{\mu\parallel}) , \\ \hat{\omega}_\mu &= \omega_\mu S^0 \equiv \frac{1}{g_B} \cos \theta_\omega V_{\mu\parallel}^{(I=0)} - \frac{1}{g} \sin \theta_\omega \bar{\Sigma} \cdot V_{\mu\perp(1)}^{(I=0)} , \\ (\hat{\omega}')_\mu &= (\omega')_\mu S^0 \equiv \frac{1}{g} \cos \theta_\omega \bar{\Sigma} \cdot V_{\mu\perp(1)}^{(I=0)} + \frac{1}{g_B} \sin \theta_\omega V_{\mu\parallel}^{(I=0)} . \end{aligned} \quad (\text{III.12})$$

The masses of these states are obtained as

$$\begin{aligned} m_{\rho,\rho'}^2 &\equiv \frac{1}{2} \left(\bar{a}_{(2)} + \bar{a}_{(3)} \mp \sqrt{(\bar{a}_{(2)} - \bar{a}_{(3)})^2 + 4\bar{a}_{(6)}^2} \right) g^2 F^2 , \\ m_{\omega,\omega'}^2 &\equiv \frac{1}{2} \left(\frac{g_B^2}{g^2} \bar{a}_{(9)} + \bar{a}_{(10)} \right. \\ &\quad \left. \mp \sqrt{\left(\frac{g_B^2}{g^2} \bar{a}_{(9)} - \bar{a}_{(10)} \right)^2 + 4 \frac{g_B^2}{g^2} \bar{a}_{(13)}^2} \right) g^2 F^2 , \end{aligned} \quad (\text{III.13})$$

where the mixing angles θ_ρ and θ_ω are determined as

$$\tan 2\theta_\rho \equiv \frac{2\bar{a}_{(6)}}{\bar{a}_{(2)} - \bar{a}_{(3)}}, \quad \tan 2\theta_\omega \equiv \frac{2g_B g \bar{a}_{(13)}}{g_B^2 \bar{a}_{(9)} - g^2 \bar{a}_{(10)}}. \quad (\text{III.14})$$

We can fix the values of ten parameters from the physical values of eight spin-1 mesons listed in Table II using Eqs. (III.10), (III.11) and (III.13) together with two conditions given in Eq. (III.6). The model still has six free parameters:

$$g, \quad g_B, \quad r_{a_1}, \quad r_{f_1}, \quad \cos \theta_\rho, \quad \cos \theta_\omega, \quad (\text{III.15})$$

which relates several interactions among the spin-1 mesons. In this paper, since we do not treat decays of the eta meson, the parameter r_{f_1} is irrelevant. So, we will determine the values of five parameters except for r_{f_1} .

In the following, we summarize the extended GT relations, the relations among one-pion decays of spin-1 mesons and the extended KSRF relations in the separated sections. To obtain some predictions analytically and numerically from them, we use experimental values in Table III.

IV. EXTENDED GOLDBERGER-TREIMAN RELATION

In this section, we investigate an extended Goldberger-Treiman relation for one-pion interactions of two different

Table II: Masses of the relevant spin-1 mesons in PDG [1]

Mesons	$I (J^{PC})$	mass (MeV)
ρ	$1 (1^{--})$	775.26 ± 0.25
ω	$0 (1^{--})$	782.65 ± 0.12
ρ'	$1 (1^{--})$	1465 ± 25
ω'	$0 (1^{--})$	$1400 - 1450$
a_1	$1 (1^{++})$	1230 ± 40
f_1	$0 (1^{++})$	1281.9 ± 0.5
b_1	$1 (1^{+-})$	1229.5 ± 3.2
h_1	$0 (1^{+-})$	1170 ± 20

Table III: Experimental values from PDG [1].

$\Gamma(\rho \rightarrow \pi\pi)$	$147.8 \pm 0.9 \text{ MeV}$
$\Gamma(\rho^0 \rightarrow e^+e^-)$	$7.04 \pm 0.06 \text{ keV}$
$\Gamma(\omega \rightarrow e^+e^-)$	$0.60 \pm 0.02 \text{ keV}$
m_{π^\pm}	$139.57018 \pm 0.00035 \text{ MeV}$
m_e	$548.57990946 \pm 0.00000022 \text{ keV}$
$\alpha \equiv \frac{e^2}{4\pi}$	$\frac{1}{137}$
f_π	$92.21 \pm 0.14 \text{ MeV}$
$\langle r^2 \rangle_V^{\pi^\pm}$	$0.452 \pm 0.011 \text{ fm}^2$

spin-1 mesons. First, we give a general discussion for the extended GT relation. Next, we derive several relations in the HLS model.

Let us start a general discussion in the case that the final spin-1 meson state has the different parity from the initial state. By requiring Lorentz covariance and parity invariance, the amplitudes of two spin-1 states coupled with the axial current j_5^α are written as^{#3}

$$\begin{aligned} \mathcal{M}^\alpha &= \int d^4x e^{-iqx} \langle V_\mu(p_2) | j_5^\alpha(x) | V_\nu(p_1) \rangle \\ &= \epsilon_\mu^*(p_2) \left[g_1(q^2) g^{\mu\nu} i p^\alpha + g_2(q^2) \frac{q^\mu q^\nu}{m_1^2 + m_2^2} i p^\alpha \right. \\ &\quad + g_3(q^2) (i q^\mu g^{\nu\alpha} + i q^\nu g^{\mu\alpha}) \\ &\quad + g_4(q^2) (i q^\mu g^{\nu\alpha} - i q^\nu g^{\mu\alpha}) \\ &\quad \left. + h_1(q^2) g^{\mu\nu} i q^\alpha + h_2(q^2) \frac{q^\mu q^\nu}{m_1^2 + m_2^2} i q^\alpha \right] \epsilon_\nu(p_1) \end{aligned} \quad (\text{IV.1})$$

where $p = \frac{p_1 + p_2}{2}$, $q = p_1 - p_2$, and $g_i(q^2)$ ($i = 1, 2, 3, 4$)

^{#3} Indices for isospin are omitted. Note that terms including the antisymmetric tensor $\epsilon^{\mu\nu\alpha\beta}$ are also allowed if the initial and final states have the same parity. In the HLS model, these contributions are obtained from the intrinsic parity odd terms, which are listed in Appendix C.

and $h_j(q^2)$ ($j = 1, 2$) are independent form factors, which are generally complex functions of q^2 . Since the axial vector current is conserved in the chiral limit, we have the Ward-Takahashi identity as $q_\alpha \mathcal{M}^\alpha = 0$, which leads to

$$\begin{aligned} & g_1(q^2) (p \cdot q) (\epsilon^* \cdot \epsilon) + g_2(q^2) (p \cdot q) \frac{(\epsilon^* \cdot q) (q \cdot \epsilon)}{m_1^2 + m_2^2} \\ & + 2g_3(q^2) (\epsilon^* \cdot q) (q \cdot \epsilon) \\ & + h_1(q^2) q^2 (\epsilon^* \cdot \epsilon) + h_2(q^2) q^2 \frac{(\epsilon^* \cdot q) (q \cdot \epsilon)}{m_1^2 + m_2^2} = 0. \quad (\text{IV.2}) \end{aligned}$$

The form factors $h_1(q^2)$ and $h_2(q^2)$ include a massless pole of the pion contribution:

$$h_n(q^2) = \frac{f_\pi}{q^2} G_{V_1 V_2 \pi}^{(n)} + \dots \quad (\text{IV.3})$$

with $n = 1, 2$. In the soft pion limit $q^2 \rightarrow 0$, the left hand side of Eq. (IV.2) is reduced to

$$\begin{aligned} & (\text{LHS of Eq. (IV.2)}) \\ & = \left[g_1(0) \frac{m_1^2 - m_2^2}{2} + f_\pi G_{V_1 V_2 \pi}^{(1)} \right] (\epsilon^* \cdot \epsilon), \quad (\text{IV.4}) \end{aligned}$$

where we used $p \cdot q = \frac{m_1^2 - m_2^2}{2}$. From this one can obtain

$$G_{V_1 V_2 \pi}^{(1)} = - \frac{m_1^2 - m_2^2}{2f_\pi} g_1(0). \quad (\text{IV.5})$$

This is an extended Goldberger-Treiman relation among a mass difference of two spin-1 mesons, their coupling to one pion and the axial form factor.

It should be noted that, if the mass splitting of the initial and final states were large, the soft pion limit would not be reasonable. The existence of the emergent symmetry in QCD implies that the mass difference of the spin-1 mesons ($\rho, a_1, \rho', \omega', b_1, f_1, h_1$) comes from the breaking of the chiral symmetry. Thus, the emergent symmetry together with the chiral symmetry ensures low energy theorems for the members of a multiplet of the symmetry.

Next, we turn to make an analysis based on the present model. In the \mathcal{L}_V part of the Lagrangian (II.28), there are no interactions among two HLS gauge fields and one pion field. However, due to the existence of the a_1 - π and f_1 - η mixings as shown in Eq. (III.8), the HLS gauge field V_μ include the fields for the physical pion in addition to the physical spin-1 mesons. Then, the interactions among two spin-1 mesons and one pion are generated from

$$\mathcal{L}_{\text{int}}^{(3)} = - \frac{1}{ig^2} \text{Tr} [(\partial_\mu V_\nu - \partial_\nu V_\mu) [V^\mu, V^\nu]] \quad (\text{IV.6})$$

included in \mathcal{L}_k of Eq. (II.29). As a result, all the interactions among two spin-1 mesons and one pion are proportional to the ratio r_{a_1}/f_π . The explicit forms of the

effective vertices are written as

$$\begin{aligned} & \Gamma^{\mu\nu} \left[(V_1)_\mu^a(p_1), (V_2)_\nu^b(p_2), \pi^c \right] \\ & = g_{V_1 V_2 \pi} \epsilon^{abc} (p_1^2 P^{\mu\nu}(p_1) - p_2^2 P^{\mu\nu}(p_2)), \\ & \Gamma^{\mu\nu} \left[(V_1)_\mu^a(p_1), (V_2^{(I=0)})_\nu(p_2), \pi^b \right] \\ & = g_{V_1 V_2 \pi} \delta^{ab} (p_1^2 P^{\mu\nu}(p_1) - p_2^2 P^{\mu\nu}(p_2)), \quad (\text{IV.7}) \end{aligned}$$

where the projection operator is defined as

$$P^{\mu\nu}(p) \equiv g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}, \quad (\text{IV.8})$$

and $g_{V_1 V_2 \pi}$ expresses the corresponding coupling:

$$\begin{aligned} g_{\rho a_1 \pi} &= g_{\rho' h_1 \pi} = \frac{r_{a_1}}{\sqrt{2} f_\pi} \cos \theta_\rho, \\ g_{\rho' a_1 \pi} &= -g_{\rho h_1 \pi} = \frac{r_{a_1}}{\sqrt{2} f_\pi} \sin \theta_\rho, \\ g_{b_1 \omega \pi} &= -\frac{r_{a_1}}{\sqrt{2} f_\pi} \sin \theta_\omega, \quad g_{b_1 \omega' \pi} = \frac{r_{a_1}}{\sqrt{2} f_\pi} \cos \theta_\omega. \end{aligned} \quad (\text{IV.9})$$

As shown in Appendix B, interactions among three spin-1 mesons including a_1 are also obtained from $\mathcal{L}_{\text{int}}^{(3)}$. The direct coupling of two spin-1 mesons with the axial external gauge field does not exist at the leading order of the present model, and only two diagrams shown in FIG. 2 contribute to the coupling to the axial vector current. The pion in Fig. 2(a) contributes to only h_1 , and the a_1 meson in Fig. 2(b) contributes to h_1 , g_1 and g_4 . We summarize their contributions in Table IV. Substituting these contributions into Eq. (IV.2) we can easily verify that the Ward-Takahashi identity is actually satisfied for any q^2 .

We next consider the soft-pion limit, $q^2 \rightarrow 0$. As expected in the general consideration given above, the pion contribution dominates over the a_1 meson contribution in h_1 . As a result, $h_1(q^2)$ is expressed as in Eq. (IV.3), where $G_{V_1 V_2 \pi}^{(1)}$ is listed in the first column of Table V. On the other hand, $g_1(0)$ is determined by taking $q^2 = 0$ limit of the a_1 meson contribution, which is listed in the second column of Table V. Since the coupling g_{a_1} in the second column is given by $g_{a_1} = -\frac{r_{a_1} m_{a_1}^2}{g}$ as shown in Appendix B, we can easily confirm that these actually satisfy the extended GT relation in Eq. (IV.5).

V. RELATIONS AMONG ONE-PION DECAYS OF SPIN-1 MESONS

In this section we give several relations among one-pion interactions of two spin-1 mesons.

We would like to stress that all the one-pion decays of spin-1 mesons are expressed by one parameter r_{a_1}/f_π reflecting the existence of the SU(4) symmetry as shown

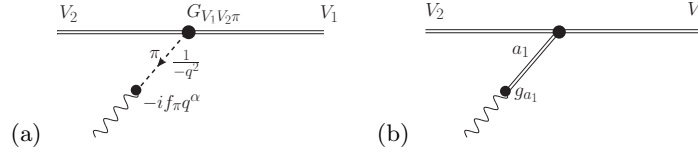


FIG. 2: Diagrams contributing to the amplitude given in Eq. (IV.1).

Table IV: Axial form factors given in the SU(4) HLS model. The function $D^{a_1}(q^2)$ is defined as $D^{a_1}(q^2) \equiv \frac{m_{a_1}^2}{m_{a_1}^2 - q^2}$. We also find that the other form factors equal to zero at the $\mathcal{O}(p^2)$ order: $h_2(q^2) = g_2(q^2) = g_3(q^2) = 0$.

	$h_1(q^2)$	$g_1(q^2)$	$g_4(q^2)$
$a_1 \rightarrow \rho$	$(m_{a_1}^2 - m_\rho^2) \left(\frac{r_{a_1}}{\sqrt{2}f_\pi} \cos \theta_\rho \right) \frac{f_\pi}{q^2} - \frac{g}{\sqrt{2}} \cos \theta_\rho \frac{g_{a_1}}{m_{a_1}^2} \frac{m_{a_1}^2 - m_\rho^2}{m_{a_1}^2} D^{a_1}(q^2)$	$\sqrt{2}g \cos \theta_\rho \frac{g_{a_1}}{m_{a_1}^2} D^{a_1}(q^2)$	$\sqrt{2}g \cos \theta_\rho \frac{g_{a_1}}{m_{a_1}^2} D^{a_1}(q^2)$
$\rho' \rightarrow a_1$	$(m_{\rho'}^2 - m_{a_1}^2) \left(\frac{r_{a_1}}{\sqrt{2}f_\pi} \sin \theta_\rho \right) \frac{f_\pi}{q^2} - \frac{g}{\sqrt{2}} \sin \theta_\rho \frac{g_{a_1}}{m_{a_1}^2} \frac{m_{\rho'}^2 - m_{a_1}^2}{m_{a_1}^2} D^{a_1}(q^2)$	$\sqrt{2}g \sin \theta_\rho \frac{g_{a_1}}{m_{a_1}^2} D^{a_1}(q^2)$	0
$h_1 \rightarrow \rho$	$-(m_{h_1}^2 - m_\rho^2) \left(-\frac{r_{a_1}}{\sqrt{2}f_\pi} \sin \theta_\rho \right) \frac{f_\pi}{q^2} - \frac{g}{\sqrt{2}} \sin \theta_\rho \frac{g_{a_1}}{m_{a_1}^2} \frac{m_{h_1}^2 - m_\rho^2}{m_{a_1}^2} D^{a_1}(q^2)$	$\sqrt{2}g \sin \theta_\rho \frac{g_{a_1}}{m_{a_1}^2} D^{a_1}(q^2)$	$-\frac{g}{\sqrt{2}} \sin \theta_\rho \frac{g_{a_1}}{m_{a_1}^2} D^{a_1}(q^2)$
$\rho' \rightarrow h_1$	$(m_{\rho'}^2 - m_{h_1}^2) \left(\frac{r_{a_1}}{\sqrt{2}f_\pi} \cos \theta_\rho \right) \frac{f_\pi}{q^2} - \frac{g}{\sqrt{2}} \cos \theta_\rho \frac{g_{a_1}}{m_{a_1}^2} \frac{m_{\rho'}^2 - m_{h_1}^2}{m_{a_1}^2} D^{a_1}(q^2)$	$\sqrt{2}g \cos \theta_\rho \frac{g_{a_1}}{m_{a_1}^2} D^{a_1}(q^2)$	$-\frac{g}{\sqrt{2}} \cos \theta_\rho \frac{g_{a_1}}{m_{a_1}^2} D^{a_1}(q^2)$
$b_1 \rightarrow \omega$	$-(m_{b_1}^2 - m_\omega^2) \left(-\frac{r_{a_1}}{\sqrt{2}f_\pi} \sin \theta_\omega \right) \frac{f_\pi}{q^2} - \frac{g}{\sqrt{2}} \sin \theta_\omega \frac{g_{a_1}}{m_{a_1}^2} \frac{m_{b_1}^2 - m_\omega^2}{m_{a_1}^2} D^{a_1}(q^2)$	$\sqrt{2}g \sin \theta_\omega \frac{g_{a_1}}{m_{a_1}^2} D^{a_1}(q^2)$	$-\frac{g}{\sqrt{2}} \sin \theta_\omega \frac{g_{a_1}}{m_{a_1}^2} D^{a_1}(q^2)$
$\omega' \rightarrow b_1$	$(m_{\omega'}^2 - m_{b_1}^2) \left(\frac{r_{a_1}}{\sqrt{2}f_\pi} \cos \theta_\omega \right) \frac{f_\pi}{q^2} - \frac{g}{\sqrt{2}} \cos \theta_\omega \frac{g_{a_1}}{m_{a_1}^2} \frac{m_{\omega'}^2 - m_{b_1}^2}{m_{a_1}^2} D^{a_1}(q^2)$	$\sqrt{2}g \cos \theta_\omega \frac{g_{a_1}}{m_{a_1}^2} D^{a_1}(q^2)$	$-\frac{g}{\sqrt{2}} \cos \theta_\omega \frac{g_{a_1}}{m_{a_1}^2} D^{a_1}(q^2)$

Table V: One pion and axial couplings in the SU(4) HLS model.

V_1	V_2	$G_{V_1 V_2 \pi}^{(1)}$	$g_1(0)$
a_1	ρ	$(m_{a_1}^2 - m_\rho^2) \left(\frac{r_{a_1}}{\sqrt{2}f_\pi} \cos \theta_\rho \right)$	$\sqrt{2}g \cos \theta_\rho \frac{g_{a_1}}{m_{a_1}^2}$
ρ'	a_1	$(m_{\rho'}^2 - m_{a_1}^2) \left(\frac{r_{a_1}}{\sqrt{2}f_\pi} \sin \theta_\rho \right)$	$\sqrt{2}g \sin \theta_\rho \frac{g_{a_1}}{m_{a_1}^2}$
h_1	ρ	$(m_{h_1}^2 - m_\rho^2) \left(\frac{r_{a_1}}{\sqrt{2}f_\pi} \sin \theta_\rho \right)$	$\sqrt{2}g \sin \theta_\rho \frac{g_{a_1}}{m_{a_1}^2}$
ρ'	h_1	$(m_{\rho'}^2 - m_{h_1}^2) \left(\frac{r_{a_1}}{\sqrt{2}f_\pi} \cos \theta_\rho \right)$	$\sqrt{2}g \cos \theta_\rho \frac{g_{a_1}}{m_{a_1}^2}$
b_1	ω	$(m_{b_1}^2 - m_\omega^2) \left(\frac{r_{a_1}}{\sqrt{2}f_\pi} \sin \theta_\omega \right)$	$\sqrt{2}g \sin \theta_\omega \frac{g_{a_1}}{m_{a_1}^2}$
ω'	b_1	$(m_{\omega'}^2 - m_{b_1}^2) \left(\frac{r_{a_1}}{\sqrt{2}f_\pi} \cos \theta_\omega \right)$	$\sqrt{2}g \cos \theta_\omega \frac{g_{a_1}}{m_{a_1}^2}$

in Eq. (IV.9). By using Eq (IV.9), the one-pion decay widths of spin-1 mesons are easily calculated:

$$\Gamma(V_i \rightarrow V_f \pi) = \frac{1}{8\kappa_{V_i \rightarrow V_f}} \frac{|\vec{p}_{V_i \rightarrow V_f}|}{\pi m_{V_i}^2} (g_{V_i V_f \pi})^2 \times (m_{V_i}^2 - m_{V_f}^2)^2 \left(3 + \frac{|\vec{p}_{V_i \rightarrow V_f}|^2}{m_{V_f}^2} \right), \quad (\text{V.1})$$

where the momentum is given as

$$|\vec{p}_{V_i \rightarrow V_f}| \equiv \frac{1}{2} \sqrt{m_{V_i}^2 - 2(m_{V_f}^2 + m_\pi^2) + \frac{(m_{V_f}^2 - m_\pi^2)^2}{m_{V_i}^2}}, \quad (\text{V.2})$$

and the factor κ depends on the isospin of the initial and final states:

$$\begin{aligned} \kappa_{h_1 \rightarrow \rho} &= \kappa_{\omega' \rightarrow b_1} = 1, \\ \kappa_{a_1 \rightarrow \rho} &= \kappa_{\rho' \rightarrow a_1} = \frac{3}{2}, \\ \kappa_{b_1 \rightarrow \omega} &= \kappa_{\rho' \rightarrow h_1} = 3. \end{aligned} \quad (\text{V.3})$$

The unknown parameter r_{a_1}/f_π in the coupling $g_{V_i V_f \pi}$ is canceled by taking ratios of these decay widths:

$$\begin{aligned} \frac{\Gamma(\rho' \rightarrow h_1 \pi)}{\Gamma(a_1 \rightarrow \rho \pi)} &= 0.16 \pm 0.07, \\ \frac{\Gamma(h_1 \rightarrow \rho \pi)}{\Gamma(\rho' \rightarrow a_1 \pi)} &= 6.4 \pm 4.3, \\ \frac{\Gamma(\rho' \rightarrow a_1 \pi)}{\Gamma(a_1 \rightarrow \rho \pi)} &= (0.15 \pm 0.11) \tan^2 \theta_\rho, \\ \frac{\Gamma(h_1 \rightarrow \rho \pi)}{\Gamma(a_1 \rightarrow \rho \pi)} &= (1.0 \pm 0.3) \tan^2 \theta_\rho, \\ \frac{\Gamma(b_1 \rightarrow \omega \pi)}{\Gamma(\omega' \rightarrow b_1 \pi)} &= (3.9 \pm 1.9) \tan^2 \theta_\omega, \end{aligned} \quad (\text{V.4})$$

where the numerical factors in the RHS are simply evaluated from the corresponding kinematical factors calculated by using the masses listed in Table II. Errors in the RHS are estimated from the errors listed in the table. Since the first two relations are independent of the parameters, experimental measurements of these ratios will check the existence of the SU(4) symmetry. Then, we can determine the mixing angles from the latter three relations.

VI. EXTENDED KSRF RELATIONS

In this section, we derive the Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin (KSRF) relations among the ρ meson mass, the $\rho\pi\pi$ coupling and the ρ -photon mixing strength, as well as their extension to the ρ' meson.

The interactions among one gauge field and two pion fields are included in the \mathcal{L}_V of the Lagrangian (II.28). Similarly to the one-pion interactions studied in the previous section, due to the existence of the a_1 - π mixing, the three point interaction $\mathcal{L}_{\text{int}}^{(3)}$ generates the interactions among a spin-1 meson and two pions. The resultant effective vertices among two pions and one vector meson are given by

$$\begin{aligned} & \Gamma^\mu [\rho_\mu^a(p), \pi^b(p_1), \pi^c(p_2)] \\ &= -i\epsilon^{abc} \left[g_{\rho\pi\pi}^{(T)}(p^2) P^{\mu\nu}(p) [p_1 - p_2]_\nu + g_{\rho\pi\pi}^{(L)}(p_1, p_2) p^\mu \right], \\ & \Gamma^\mu [(\rho')_\mu^a(p), \pi^b(p_1), \pi^c(p_2)] \\ &= -i\epsilon^{abc} \left[g_{\rho'\pi\pi}^{(T)}(p^2) P^{\mu\nu}(p) [p_1 - p_2]_\nu + g_{\rho'\pi\pi}^{(L)}(p_1, p_2) p^\mu \right] \end{aligned} \quad (\text{VI.1})$$

where $p = p_1 + p_2$, and

$$\begin{aligned} g_{\rho\pi\pi}^{(T)}(p^2) &= \frac{m_\rho^2 - r_{a_1}^2 p^2}{\sqrt{2} g f_\pi^2} \cos \theta_\rho, \\ g_{\rho'\pi\pi}^{(T)}(p^2) &= \frac{m_{\rho'}^2 - r_{a_1}^2 p^2}{\sqrt{2} g f_\pi^2} \sin \theta_\rho, \\ g_{\rho\pi\pi}^{(L)}(p_1, p_2) &= \frac{m_\rho^2 \cos \theta_\rho}{\sqrt{2} g f_\pi^2} \frac{(p_2)^2 - (p_1)^2}{(p_1 + p_2)^2}, \\ g_{\rho'\pi\pi}^{(L)}(p_1, p_2) &= \frac{m_{\rho'}^2 \sin \theta_\rho}{\sqrt{2} g f_\pi^2} \frac{(p_2)^2 - (p_1)^2}{(p_1 + p_2)^2}. \end{aligned} \quad (\text{VI.2})$$

Since $g_{V\pi\pi}^{(L)}(p_1, p_2)$ ($V = \rho, \rho'$) vanishes for on-shell pion, only $g_{V\pi\pi}^{(T)}(p^2)$ is relevant for $V \rightarrow \pi\pi$ decay and the electromagnetic form factor of pion. Furthermore, when the vector mesons are on their mass shell, the ratio of two $V\pi\pi$ couplings is related to the mixing angle as

$$\frac{g_{\rho'\pi\pi}^{(T)}(m_{\rho'}^2)}{g_{\rho\pi\pi}^{(T)}(m_\rho^2)} = \frac{m_{\rho'}^2}{m_\rho^2} \tan \theta_\rho. \quad (\text{VI.3})$$

We introduce the photon field A_μ by replacing the external gauge field as

$$\mathcal{V}_\mu = e A_\mu Q, \quad (\text{VI.4})$$

where e is the electromagnetic coupling constant, and

$$Q = \sqrt{2} \left(S^3 + \frac{1}{3} S^0 \right) = \begin{pmatrix} t^3 + \frac{1}{6} 1_2 & 0 \\ 0 & t^3 + \frac{1}{6} 1_2 \end{pmatrix}. \quad (\text{VI.5})$$

The \mathcal{L}_V part of the Lagrangian generates the mixing between a vector meson and the photon. The mixing strengths for ρ, ρ', ω and ω' mesons are expressed as

$$g_\rho \equiv \frac{\sqrt{2} m_\rho^2 \cos \theta_\rho}{g}, \quad g_{\rho'} \equiv \frac{\sqrt{2} m_{\rho'}^2 \sin \theta_\rho}{g}, \quad (\text{VI.6})$$

$$g_\omega \equiv \frac{\sqrt{2} m_\omega^2 \cos \theta_\omega}{3g_B}, \quad g_{\omega'} \equiv \frac{\sqrt{2} m_{\omega'}^2 \sin \theta_\omega}{3g_B}. \quad (\text{VI.7})$$

Similarly to Eq. (VI.3), several ratios of two of above quantities are expressed as

$$\begin{aligned} \frac{g_{\rho'}}{g_\rho} &= \frac{m_{\rho'}^2}{m_\rho^2} \tan \theta_\rho, \\ \frac{g_\omega}{g_\rho} &= \frac{1}{3} \frac{m_\omega^2}{m_\rho^2} \frac{g}{g_B} \frac{\cos \theta_\omega}{\cos \theta_\rho}, \\ \frac{g_{\omega'}}{g_\omega} &= \frac{m_{\omega'}^2}{m_\omega^2} \tan \theta_\omega, \end{aligned} \quad (\text{VI.8})$$

Now, comparing the expressions in Eq. (VI.6) with the two-pion vertices in Eq. (VI.2), one can find the KSRF I relation and the extended one for ρ' in the soft momentum limit, $p = 0$:

$$g_\rho = 2g_{\rho\pi\pi}^{(T)}(p^2 = 0) f_\pi^2, \quad g_{\rho'} = 2g_{\rho'\pi\pi}^{(T)}(p^2 = 0) f_\pi^2. \quad (\text{VI.9})$$

On the other hand, for the on-shell vector mesons, they become

$$\frac{2g_{\rho\pi\pi}^{(T)}(m_\rho^2) f_\pi^2}{g_\rho} = \frac{2g_{\rho'\pi\pi}^{(T)}(m_{\rho'}^2) f_\pi^2}{g_{\rho'}} = 1 - r_{a_1}^2. \quad (\text{VI.10})$$

This implies that the deviation for the on-shell ρ from the KSRF I relation is caused by the term including r_{a_1} , which is generated from $\mathcal{L}_{\text{int}}^{(3)}$, in $g_{\rho\pi\pi}^{(T)}(p^2)$.

Let us consider the relations among several relevant decay widths. The decay widths for the $\rho \rightarrow \pi\pi$ and $\rho^0 \rightarrow e^+e^-$ are calculated as

$$\Gamma(\rho \rightarrow \pi\pi) = \frac{1}{6\pi m_\rho^2} \left[\frac{m_\rho^2 - 4m_\pi^2}{4} \right]^{\frac{3}{2}} \left| g_{\rho\pi\pi}^{(T)}(m_\rho^2) \right|^2, \quad (\text{VI.11})$$

$$\Gamma(\rho^0 \rightarrow e^+e^-) = \frac{4\pi\alpha^2}{3} \left| \frac{g_\rho}{m_\rho^2} \right|^2 \frac{m_\rho^2 + 2m_e^2}{m_\rho^2} \sqrt{m_\rho^2 - 4m_e^2}, \quad (\text{VI.12})$$

and similarly for $\rho' \rightarrow \pi\pi$ and $\rho', \omega, \omega' \rightarrow e^+e^-$. Combining the relations in Eq. (VI.3) and (VI.8), we obtain

$$\frac{\Gamma(\rho' \rightarrow \pi\pi)}{\Gamma(\rho \rightarrow \pi\pi)} = (28 \pm 2) \tan^2 \theta_\rho, \quad (\text{VI.13})$$

$$\frac{\Gamma(\rho'^0 \rightarrow e^+e^-)}{\Gamma(\rho^0 \rightarrow e^+e^-)} = (1.89 \pm 0.03) \tan^2 \theta_\rho, \quad (\text{VI.14})$$

$$\frac{\Gamma(\omega \rightarrow e^+e^-)}{\Gamma(\rho^0 \rightarrow e^+e^-)} = (0.112 \pm 0.000) \frac{g^2 \cos^2 \theta_\omega}{g_B^2 \cos^2 \theta_\rho}, \quad (\text{VI.15})$$

$$\frac{\Gamma(\omega' \rightarrow e^+e^-)}{\Gamma(\omega \rightarrow e^+e^-)} = (1.01 \pm 0.03) \tan^2 \theta_\omega, \quad (\text{VI.16})$$

where 0.000 in the third equations implies that the error is smaller than 0.0005. Taking the ratio of Eq. (VI.13) and Eq. (VI.14), we obtain the following parameter free relation:

$$\frac{\Gamma(\rho' \rightarrow \pi\pi)}{\Gamma(\rho \rightarrow \pi\pi)} \frac{\Gamma(\rho'^0 \rightarrow e^+e^-)}{\Gamma(\rho^0 \rightarrow e^+e^-)} = (15 \pm 1), \quad (\text{VI.17})$$

which is regarded as an experimental check of the existence of SU(4) symmetry for spin-1 mesons.

The relations in Eqs. (VI.13)-(VI.16) can be used to determine the relevant model parameters. At this moment, we can set up an upper limit for the mixing angle $\tan^2 \theta_\rho$ in the following way: The total decay width of ρ' and the partial decay width of the $\rho \rightarrow \pi\pi$ channel are known as $\Gamma^{\text{total}}(\rho') = 400 \pm 60 \text{ MeV}$ and $\Gamma(\rho \rightarrow \pi\pi) = 147.8 \text{ MeV}$, respectively. Then the upper limit of the LHS of Eq. (VI.13) is estimated as

$$\frac{\Gamma(\rho' \rightarrow \pi\pi)}{\Gamma(\rho \rightarrow \pi\pi)} \leq \frac{\Gamma^{\text{total}}(\rho')}{\Gamma(\rho \rightarrow \pi\pi)} \sim 3. \quad (\text{VI.18})$$

From this together with Eq. (VI.13), the limit is obtained as

$$\tan^2 \theta_\rho \lesssim 0.1, \quad (\text{VI.19})$$

which implies that the mixing between V_\parallel and $V_{\perp(1)}$ is not large. From this upper limit for the mixing angle, the ratio of e^+e^- decays of ρ' and ρ mesons has an upper limit as

$$\frac{\Gamma(\rho'^0 \rightarrow e^+e^-)}{\Gamma(\rho^0 \rightarrow e^+e^-)} \lesssim 0.2. \quad (\text{VI.20})$$

Using the upper limit for $\tan^2 \theta_\rho$ in Eqs. (VI.19), we obtain the upper limits for the ratios of one-pion decays of spin-1 mesons:

$$\begin{aligned} \frac{\Gamma(\rho' \rightarrow a_1\pi)}{\Gamma(a_1 \rightarrow \rho\pi)} &\lesssim 0.02, \\ \frac{\Gamma(h_1 \rightarrow \rho\pi)}{\Gamma(a_1 \rightarrow \rho\pi)} &\lesssim 0.1, \end{aligned} \quad (\text{VI.21})$$

which can be tested in future experiments.

Next, we consider a constraint from Eq. (VI.15). Using the experimental value $\Gamma(\omega \rightarrow e^+e^-) = 0.60 \pm 0.02 \text{ keV}$, we obtain

$$\frac{g^2 \cos^2 \theta_\omega}{g_B^2 \cos^2 \theta_\rho} = 0.76 \pm 0.03. \quad (\text{VI.22})$$

When the gauge couplings for SU(4) HLS and U(1) HLS are equal to each other, this together with Eq. (VI.19) gives a constraint as

$$\tan^2 \theta_\omega \lesssim 0.5, \quad (\text{VI.23})$$

and

$$\begin{aligned} \frac{\Gamma(b_1 \rightarrow \omega\pi)}{\Gamma(\omega' \rightarrow b_1\pi)} &\lesssim 2, \\ \frac{\Gamma(\omega' \rightarrow e^+e^-)}{\Gamma(\omega \rightarrow e^+e^-)} &\lesssim 0.5. \end{aligned} \quad (\text{VI.24})$$

VII. NUMERICAL ANALYSIS

In this section, we determine the model parameters from the relevant experimental data, and make several phenomenological predictions.

We first construct the electromagnetic form factor of pion. From the second term in the last line of Eq. (B.3), we can read the direct $\gamma\pi\pi$ coupling as

$$g_{\gamma\pi\pi} \equiv 1 - \frac{m_\rho^2 \cos^2 \theta_\rho + m_{\rho'}^2 \sin^2 \theta_\rho}{g^2 f_\pi^2}. \quad (\text{VII.1})$$

From this and Eqs. (VI.2) and (VI.6), the pion space-like form factor is given by

$$F_V^{\pi^\pm}(Q^2) = g_{\gamma\pi\pi} + \frac{g_\rho g_{\rho\pi\pi}^{(T)}(-Q^2)}{m_\rho^2 + Q^2} + \frac{g_{\rho'} g_{\rho'\pi\pi}^{(T)}(-Q^2)}{m_{\rho'}^2 + Q^2} \quad (\text{VII.2})$$

with $Q^2 = -q^2$, where q is the photon momentum. This form factor is normalized as $F_V^{\pi^\pm}(Q^2 = 0) = 1$ reflecting the existence of the electromagnetic U(1) symmetry. From this, the pion charge radius is calculated as

$$\langle r^2 \rangle_V^{\pi^\pm} \equiv -6 \frac{\partial F_V^{\pi^\pm}(Q^2)}{\partial Q^2} \bigg|_{Q^2=0} = 6 \frac{1 - r_{a_1}^2}{g^2 f_\pi^2}. \quad (\text{VII.3})$$

By using Eqs. (VI.6), (VI.10), and (VII.3), the param-

eters are expressed as

$$|r_{a_1}| = \sqrt{1 - \left(\frac{2g_{\rho\pi\pi}^{(T)}(m_\rho^2)f_\pi^2}{g_\rho} \right)}, \quad (\text{VII.4})$$

$$|g| = \frac{m_\rho}{f_\pi} \sqrt{\left(\frac{2g_{\rho\pi\pi}^{(T)}(m_\rho^2)f_\pi^2}{g_\rho} \right) / \left(\frac{1}{6} \langle r^2 \rangle_V^{\pi^\pm} m_\rho^2 \right)}, \quad (\text{VII.5})$$

$$\tan^2 \theta_\rho = \frac{\left(\frac{1}{6} \langle r^2 \rangle_V^{\pi^\pm} m_\rho^2 \right)}{\left(\frac{2g_{\rho\pi\pi}^{(T)}(m_\rho^2)f_\pi^2}{g_\rho} \right) \left(\frac{g_\rho}{\sqrt{2}m_\rho f_\pi} \right)^2} - 1. \quad (\text{VII.6})$$

We should note that, since the pion charge radius is positive and

$$\frac{2g_{\rho\pi\pi}^{(T)}(m_\rho^2)f_\pi^2}{g_\rho} = \frac{g^2 f_\pi^2}{6 \langle r^2 \rangle_V^{\pi^\pm}} \quad (\text{VII.7})$$

is satisfied, one can find that the couplings $g_{\rho\pi\pi}^{(T)}(m_\rho^2)$ and g_ρ have the same sign. Then, the inside of the square root in the RHS of Eq. (VII.5) is always positive.

Substituting the experimental values listed in Table III into Eqs. (VII.4)-(VII.6), we have

$$\begin{aligned} |r_{a_1}| &= 0.41 \pm 0.10, \\ |g| &= 7.1 \pm 0.5, \\ \tan^2 \theta_\rho &= -0.03 \pm 0.25 \end{aligned} \quad (\text{VII.8})$$

where we added 10% errors expected from higher order corrections [11]. Then the upper limit of $\tan^2 \theta_\rho$, $\tan^2 \theta_\rho \lesssim 0.1$ given in Eq. (VI.19), is within the errors of above determination. The electromagnetic form factor obtained from these values is shown in Fig. 3 together with the experimental data. This shows that the predicted form factor reasonably reproduce the experimental data, taking account of their errors.

We would like to note that the both numerators of the ρ and ρ' contributions in Eq. (VII.2) have the same sign, which is contrasted to the result by a holographic QCD model [17]. Furthermore, the direct $\gamma\pi\pi$ coupling is evaluated through

$$g_{\gamma\pi\pi} = 1 - \frac{m_{\rho'}^2}{m_\rho^2} \frac{\left(\frac{1}{6} \langle r^2 \rangle_V^{\pi^\pm} m_\rho^2 \right)}{\left(\frac{2g_{\rho\pi\pi}^{(T)}(m_\rho^2)f_\pi^2}{g_\rho} \right)} + \left(\frac{g_\rho}{\sqrt{2}m_\rho f_\pi} \right)^2 \left(\frac{m_{\rho'}^2}{m_\rho^2} - 1 \right) \quad (\text{VII.9})$$

as $g_{\gamma\pi\pi} = -0.28 \pm 0.13$ by using the values in Table III. This result means that there is a slight deviation from the vector meson dominance.

At the end of this section, we estimate several decay widths of spin-1 mesons by using the parameter set given in Eqs. (VII.8) and show them in Table VI.

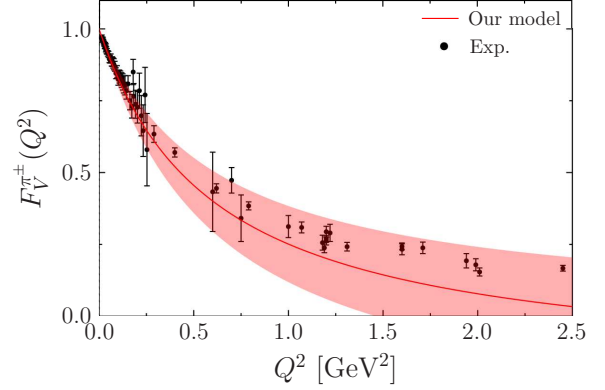


FIG. 3: Electromagnetic form factor of pion. Predicted form factor from the central values of the parameters in Eq. (VII.8) is shown by the red curve. Shaded area shows the errors of the parameters. Black dots express the experimental data given in Refs. [12–16].

Table VI: Predicted values of one-pion decay widths of spin-1 mesons estimated from the parameter set in Eq. (VII.8).

Decay mode	Partial width (MeV)
$\Gamma(a_1 \rightarrow \rho\pi)$	470 ± 400
$\Gamma(\rho' \rightarrow a_1\pi)$	< 50
$\Gamma(h_1 \rightarrow \rho\pi)$	< 60
$\Gamma(\rho' \rightarrow h_1\pi)$	80 ± 300

VIII. SUMMARY AND DISCUSSIONS

We constructed a chiral Lagrangian with an $SU(4) \times U(1)$ hidden local symmetry which includes the spin-1 mesons, $(\rho, a_1, \rho', \omega', b_1, f_1, h_1)$, together with pion. We found that each coupling of the interaction among one pion and two spin-1 mesons is proportional to the mass difference of the relevant spin-1 mesons similarly to the Goldberger-Treiman relation. In addition, there were the relations among one-pion decays of spin-1 mesons thanks to the existence of the $SU(4)$ emergent symmetry. Furthermore, we found a relation among the mass of ρ' meson, the $\rho'\pi\pi$ coupling and the ρ' -photon mixing strength as well as the Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin relation for the ρ meson. We summarize these predictions in Table VII.

Using two ratios indicated in Table VII together with the total widths of ρ' and ω' mesons, we obtained several upper limit for the ratios as shown in the last column of the table. By testing them in future experiments, we can verify the existence of the emergent symmetry.

There also exist hadronic decays which involve the intrinsic parity odd terms shown in Appendix C. Possible decay modes are expressed by “ \times ” in Table VIII, such as

Table VII: Predictions obtained from the $SU(4) \times U(1)$ HLS model.

Independent of the parameters			
$\Gamma(\rho' \rightarrow h_1\pi)$	=	0.16 ± 0.07	
$\Gamma(a_1 \rightarrow \rho\pi)$			
$\Gamma(h_1 \rightarrow \rho\pi)$	=	6.4 ± 4.3	
$\Gamma(\rho' \rightarrow a_1\pi)$			
Dependent on θ_ρ or θ_ω			
$\Gamma(\rho' \rightarrow a_1\pi)$	=	$(0.15 \pm 0.11) \tan^2 \theta_\rho$	$\lesssim 0.015$
$\Gamma(a_1 \rightarrow \rho\pi)$			
$\Gamma(h_1 \rightarrow \rho\pi)$	=	$(1.0 \pm 0.3) \tan^2 \theta_\rho$	$\lesssim 0.10$
$\Gamma(a_1 \rightarrow \rho\pi)$			
$\Gamma(\rho' \rightarrow \pi\pi)$	=	$(28 \pm 2) \tan^2 \theta_\rho$	(Input)
$\Gamma(\rho \rightarrow \pi\pi)$			
$\Gamma(\rho'^0 \rightarrow e^+e^-)$	=	$(1.89 \pm 0.03) \tan^2 \theta_\rho$	$\lesssim 0.2$
$\Gamma(\rho^0 \rightarrow e^+e^-)$			
$\Gamma(\omega \rightarrow e^+e^-)$	=	$(0.11219 \pm 0.00004) \frac{g^2}{g_B^2} \frac{\cos^2 \theta_\omega}{\cos^2 \theta_\rho}$	(Input)
$\Gamma(\rho^0 \rightarrow e^+e^-)$			
$\Gamma(b_1 \rightarrow \omega\pi)$	=	$(3.9 \pm 1.9) \tan^2 \theta_\omega$	
$\Gamma(\omega' \rightarrow b_1\pi)$			
$\Gamma(\omega' \rightarrow e^+e^-)$	=	$(1.01 \pm 0.03) \tan^2 \theta_\omega$	
$\Gamma(\omega \rightarrow e^+e^-)$			

$\rho' \rightarrow \omega\pi$ and $\omega \rightarrow 3\pi$. We listed the allowed operators in Appendix C.

Similarly to the result obtained in the generalized HLS at $\mathcal{O}(p^2)$ [9, 10], we also found $\Gamma(a_1 \rightarrow \pi\gamma) = 0$ together with $\Gamma(b_1 \rightarrow \pi\gamma) = 0$ and $\Gamma(h_1 \rightarrow \pi\gamma) = 0$ at the leading order, as we showed some detail calculations in Appendix D. As in the case of the generalized HLS, we expect that non-vanishing contributions will be produced by higher order correction of the derivative expansion [9, 10, 18].

We have to remark that our analysis are done in the chiral broken phase since we used the nonlinear realization of the chiral symmetry. On the other hand, the existence of the emergent symmetry is proposed by reducing the Dirac zero mode in the lattice QCD, which corresponds to remove the dominant contribution of the chiral symmetry breaking as shown by the Banks-Casher relation. It will be interesting to clarify the correspondence between our model and the lattice QCD result, which we leave for future works.

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Table VIII: Two-body pionic decay channels. “–” implies that the decay mode is prohibited kinematically or by the isospin, \mathcal{P} , and \mathcal{C} while “ \times ” and “ \checkmark ” mean that such decay channel is allowed through the operators with the intrinsic parity odd and even, respectively.

Initial	mass (MeV)	channel											
		$\pi\pi$	$\rho\pi$	$\omega\pi$	$\rho'\pi$	$\omega'\pi$	$a_1\pi$	$f_1\pi$	$b_1\pi$	$h_1\pi$	3π	$\eta\pi\pi$	4π
ρ	775.26	\checkmark	–	–	–	–	–	–	–	–	–	–	\checkmark
ω	782.65	–	–	–	–	–	–	–	–	–	\times	–	–
ρ'	1465	\checkmark	–	\times	–	–	\checkmark	–	–	\checkmark	–	\times	\checkmark
ω'	~ 1425	–	\times	–	–	–	–	–	\checkmark	–	\times	–	–
a_1	1230	–	\checkmark	–	–	–	–	–	–	–	\checkmark	–	–
f_1	1281.9	–	–	–	–	–	–	–	–	–	–	\checkmark	\times
b_1	1229.5	–	–	\checkmark	–	–	–	–	–	–	–	\checkmark	\times
h_1	1170	–	\checkmark	–	–	–	–	–	–	–	\checkmark	–	–

Appendix A: Generators of an $SU(4)$

The generators of an $SU(4)$ are defined as $T^A = \{S^a, X_{(3)}^a, X_{(1)}^a, X_{(2)}^a, X_{(3)}^0, X_{(1)}^0, X_{(2)}^0\}$ with $A = 1, \dots, 15$ and $a = 1, 2, 3$:

$$\begin{aligned}
S^a &\equiv \frac{1}{\sqrt{2}} \begin{pmatrix} t^a & 0 \\ 0 & t^a \end{pmatrix}, \\
X_{(3)}^a &\equiv \frac{1}{\sqrt{2}} \begin{pmatrix} t^a & 0 \\ 0 & -t^a \end{pmatrix}, \\
X_{(1)}^a &\equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & t^a \\ t^a & 0 \end{pmatrix}, \\
X_{(2)}^a &\equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -it^a \\ it^a & 0 \end{pmatrix}, \\
X_{(3)}^0 &\equiv \frac{1}{2\sqrt{2}} \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \\
X_{(1)}^0 &\equiv \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} = \frac{1}{2\sqrt{2}} \bar{\Sigma}, \\
X_{(2)}^0 &\equiv \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & -i1_2 \\ i1_2 & 0 \end{pmatrix}
\end{aligned} \tag{A.1}$$

where $a = 1, 2, 3$ and $t^a = \sigma^a/2$. An $SU(4)$ group includes its subgroup $SU(2)$ whose generator is S^a . The generator of a $U(1)$ is also defined as

$$S^0 \equiv \frac{1}{2\sqrt{2}} \begin{pmatrix} 1_2 & 0 \\ 0 & 1_2 \end{pmatrix}. \tag{A.2}$$

The commutation relation between the generators are

obtained as

$$\begin{aligned}
[X_{(i)}^a, X_{(j)}^b] &= \frac{1}{\sqrt{2}} i \epsilon^{ijk} \delta^{ab} X_{(k)}^0 + \frac{1}{\sqrt{2}} i \epsilon^{abc} \delta_{ij} S^c, & [S^a, X_{(i)}^0] &= 0, \\
[X_{(i)}^a, X_{(j)}^0] &= \frac{1}{\sqrt{2}} i \epsilon^{ijk} X_{(k)}^a, & [S^a, X_{(i)}^b] &= \frac{1}{\sqrt{2}} i \epsilon^{abc} X_{(i)}^c, \\
[X_{(i)}^0, X_{(j)}^0] &= \frac{1}{\sqrt{2}} i \epsilon^{ijk} X_{(k)}^0, & [S^a, S^b] &= \frac{1}{\sqrt{2}} i \epsilon^{abc} S^c.
\end{aligned} \tag{A.3}$$

Appendix B: Detailed calculations of Lagrangian terms

In this appendix we show detailed calculations for obtaining the interaction terms among the spin-1 mesons and pion.

We expand the Maurer-Cartan 1-forms given in Eq. (II.13) in the unitary gauge $p = s = \tilde{s} = 0$:

$$\begin{aligned}
\hat{\alpha}_{\mu\parallel}(x) &= -\frac{1}{2iF_\pi} [\pi, \partial_\mu \pi] - V_{\mu\parallel} + 2\text{Tr} [\Xi(\pi) \cdot \mathcal{V}_\mu \cdot \Xi^\dagger(\pi) \cdot S^a] S^a + \dots, \\
\hat{\alpha}_{\mu\perp(3)}(x) &= \frac{1}{F_\pi} \partial_\mu \pi - \frac{1}{6F_\pi^3} [\pi, [\pi, \partial_\mu \pi]] - V_{\mu\perp(3)} + 2\text{Tr} [\Xi(\pi) \cdot \mathcal{V}_\mu \cdot \Xi^\dagger(\pi) \cdot X_{\perp(3)}^a] X_{\perp(3)}^a + \dots, \\
\hat{\alpha}_{\mu\perp(1)}^{(m)}(x) &= V_{\mu\perp(1)}, \quad \hat{\alpha}_{\mu\perp(2)}^{(m)}(x) = V_{\mu\perp(2)}, \quad \hat{\alpha}_{\mu\perp(3)}^{(m)}(x) = V_{\mu\perp(3)}.
\end{aligned} \tag{B.1}$$

with $V_{\mu\parallel} = 2\text{Tr} [V_\mu \cdot S^a] S^a$ and $V_{\mu\perp(i)} = 2\text{Tr} [V_\mu \cdot X_{(i)}^a] X_{(i)}^a$. By using Eq. (B.1), \mathcal{L}_V is written as

$$\begin{aligned}
\mathcal{L}_V &= \bar{a}_{(1)} F^2 \text{Tr} \left[\left(\frac{1}{F_\pi} \partial_\mu \pi - \frac{1}{6F_\pi^3} [\pi, [\pi, \partial_\mu \pi]] + 2\text{Tr} [\Xi(\pi) \cdot \mathcal{V}_\mu \cdot \Xi^\dagger(\pi) \cdot X_{\perp(3)}^a] X_{\perp(3)}^a \right)^2 \right] \\
&+ \bar{a}_{(2)} F^2 \text{Tr} \left[\left(\frac{1}{2iF_\pi} [\pi, \partial_\mu \pi] - V_{\mu\parallel} + 2\text{Tr} [\Xi(\pi) \cdot \mathcal{V}_\mu \cdot \Xi^\dagger(\pi) \cdot S^a] S^a \right)^2 \right] \\
&+ \bar{a}_{(3)} F^2 \text{Tr} [(V_{\mu\perp(1)})^2] + \bar{a}_{(4)} F^2 \text{Tr} [(V_{\mu\perp(2)})^2] + \bar{a}_{(5)} F^2 \text{Tr} [(V_{\mu\perp(3)})^2] \\
&+ \bar{a}_{(6)} 2F^2 \text{Tr} \left[\left(\frac{1}{2iF_\pi} [\pi, \partial_\mu \pi] - V_{\mu\parallel} + 2\text{Tr} [\Xi(\pi) \cdot \mathcal{V}_\mu \cdot \Xi^\dagger(\pi) \cdot S^a] S^a \right) \cdot V_{\perp(1)}^\mu \cdot \bar{\Sigma} \right] \\
&- \bar{a}_{(7)} 2F^2 \text{Tr} \left[\left(\frac{1}{F_\pi} \partial_\mu \pi - \frac{1}{6F_\pi^3} [\pi, [\pi, \partial_\mu \pi]] + 2\text{Tr} [\Xi(\pi) \cdot \mathcal{V}_\mu \cdot \Xi^\dagger(\pi) \cdot X_{\perp(3)}^a] X_{\perp(3)}^a \right) \cdot V_{\perp(3)}^\mu \right] \\
&+ [\text{Terms for } (I=0) : \text{from } \bar{a}_{(8)} \text{ to } \bar{a}_{(14)}] + \dots.
\end{aligned} \tag{B.2}$$

Substituting the masses and the eigenstates defined in Sec. III into Eq. (B.2), we have

$$\begin{aligned}
\mathcal{L}_V &= \text{Tr} [\partial_\mu \pi]^2 + m_\rho^2 \text{Tr} [\rho_\mu]^2 + m_{\rho'}^2 \text{Tr} [(\rho')_\mu]^2 + m_{a_1}^2 \text{Tr} [(a_1)_\mu]^2 + m_{b_1}^2 \text{Tr} [(b_1)_\mu]^2 \\
&+ m_\omega^2 \text{Tr} [\omega_\mu]^2 + m_{\omega'}^2 \text{Tr} [(\omega')_\mu]^2 + m_{f_1}^2 \text{Tr} [(f_1)_\mu]^2 + m_{h_1}^2 \text{Tr} [(h_1)_\mu]^2 \\
&- \left(\frac{1}{3f_\pi^2} - \frac{m_{\rho'}^2 \sin^2 \theta_\rho + m_\rho^2 \cos^2 \theta_\rho}{4g^2 f_\pi^4} \right) \frac{4}{i^2} \text{Tr} [[\pi, \partial_\mu \pi] [\pi, \partial^\mu \pi]] + \left(\frac{r_{a_1} m_{a_1}^2}{6g f_\pi^3} \right) \frac{4}{i^2} \text{Tr} [[\pi, \partial_\mu \pi] \cdot [\pi, (a_1)^\mu]] \\
&+ \left(\frac{m_\rho^2 \cos \theta_\rho}{\sqrt{2} g f_\pi^2} \right) \frac{2\sqrt{2}}{i} \text{Tr} [[\pi, \partial_\mu \pi] \cdot \rho^\mu] + \left(\frac{m_{\rho'}^2 \sin \theta_\rho}{\sqrt{2} g f_\pi^2} \right) \frac{2\sqrt{2}}{i} \text{Tr} [[\pi, \partial_\mu \pi] \cdot (\rho')^\mu] \\
&- \left(\frac{\sqrt{2} m_\rho^2 \cos \theta_\rho}{g} \right) 2\text{Tr} [\mathcal{V}_{\mu\parallel} \cdot \rho^\mu] - \left(\frac{\sqrt{2} m_{\rho'}^2 \sin \theta_\rho}{g} \right) 2\text{Tr} [\mathcal{V}_{\mu\parallel} \cdot (\rho')^\mu] \\
&- \left(\frac{\sqrt{2} m_\omega^2 \cos \theta_\omega}{3g_B} \right) 6\text{Tr} [\mathcal{V}_\mu^0 \cdot \omega^\mu] - \left(\frac{\sqrt{2} m_{\omega'}^2 \sin \theta_\omega}{3g_B} \right) 6\text{Tr} [\mathcal{V}_\mu^0 \cdot (\omega')^\mu] \\
&+ f_\pi 2\text{Tr} [\mathcal{A}_\mu (\partial^\mu \pi)] + f_\pi 2\text{Tr} [\mathcal{A}_\mu^0 (\partial^\mu \pi)] - \left(\frac{r_{a_1} m_{a_1}^2}{g} \right) 2\text{Tr} [\mathcal{A}_\mu \cdot (a_1)^\mu] - \left(\frac{r_{a_1} m_{a_1}^2}{g} \right) 2\text{Tr} [\mathcal{A}_\mu^0 \cdot (f_1)^\mu]
\end{aligned}$$

$$\begin{aligned}
& - \left(\frac{\sqrt{2}m_\rho^2 \cos \theta_\rho}{gf_\pi} \right) \frac{2\sqrt{2}}{i} \text{Tr} [\mathcal{A}_\mu \cdot [\pi, \rho^\mu]] - \left(\frac{\sqrt{2}m_{\rho'}^2 \sin \theta_\rho}{gf_\pi} \right) \frac{2\sqrt{2}}{i} \text{Tr} [\mathcal{A}_\mu \cdot [\pi, (\rho')^\mu]] \\
& - \left(\frac{r_{a_1} m_{a_1}^2}{gf_\pi} \right) \frac{2\sqrt{2}}{i} \text{Tr} [\mathcal{V}_{\mu\parallel} \cdot [\pi, (a_1)^\mu]] + \left(1 - \frac{m_{\rho'}^2 \sin^2 \theta_\rho + m_\rho^2 \cos^2 \theta_\rho}{g^2 f_\pi^2} \right) \frac{2\sqrt{2}}{i} \text{Tr} [\mathcal{V}_{\mu\parallel} \cdot [\pi, \partial^\mu \pi]] + \dots, \quad (\text{B.3})
\end{aligned}$$

where the vector and axial parts of the external gauge field are defined as

$$\begin{aligned}
\mathcal{V}_{\mu\parallel} &= \mathcal{V}_{\mu\parallel}^a \cdot S^a = \frac{1}{2} (\mathcal{R}_\mu^a + \mathcal{L}_\mu^a) \cdot S^a, \\
\mathcal{A}_\mu &= \mathcal{A}_\mu^a \cdot X_{\perp(3)}^a = \frac{1}{2} (\mathcal{R}_\mu^a - \mathcal{L}_\mu^a) \cdot X_{\perp(3)}^a
\end{aligned} \quad (\text{B.4})$$

by using the external fields given in Eq. (II.11). Here Eq. (II.11) is rewritten as

$$\mathcal{V}_\mu = \sqrt{2} \left(\mathcal{V}_{\mu\parallel} + \mathcal{A}_\mu + \mathcal{V}_\mu^0 S^0 + \mathcal{A}_\mu^0 X_{\perp(3)}^0 \right). \quad (\text{B.5})$$

On the other hand, the kinetic term of the HLS gauge field also gives three point interaction terms:

$$\begin{aligned}
\mathcal{L}_{\text{int}}^{(3)} &= -\frac{1}{ig^2} \text{Tr} [(\partial_\mu V_\nu - \partial_\nu V_\mu) [V^\mu, V^\nu]] \\
&= -\frac{1}{2\sqrt{2}g^2} \epsilon^{abc} \left(\partial_\mu V_{\nu\parallel}^a - \partial_\nu V_{\mu\parallel}^a \right) V_{\parallel}^{\mu b} V_{\parallel}^{\nu c} - \frac{1}{2\sqrt{2}g^2} \epsilon^{abc} \left(\partial_\mu V_{\nu\parallel}^a - \partial_\nu V_{\mu\parallel}^a \right) V_{\perp(i)}^{\mu b} V_{\perp(i)}^{\nu c} \\
&\quad - \frac{1}{\sqrt{2}g^2} \epsilon^{ijk} \left(\partial_\mu V_{\nu\perp(i)}^a - \partial_\nu V_{\mu\perp(i)}^a \right) V_{\perp(j)}^{\mu a} V_{\perp(k)}^{\nu(I=0)} - \frac{1}{\sqrt{2}g^2} \epsilon^{abc} \left(\partial_\mu V_{\nu\perp(i)}^a - \partial_\nu V_{\mu\perp(i)}^a \right) V_{\parallel}^{\mu b} V_{\perp(i)}^{\nu c} \\
&\quad - \frac{1}{2\sqrt{2}g^2} \epsilon^{ijk} \left(\partial_\mu V_{\nu\perp(i)}^{(I=0)} - \partial_\nu V_{\mu\perp(i)}^{(I=0)} \right) V_{\perp(j)}^{\mu a} V_{\perp(k)}^{\nu a} - \frac{1}{2\sqrt{2}g^2} \epsilon^{ijk} \left(\partial_\mu V_{\nu\perp(i)}^{(I=0)} - \partial_\nu V_{\mu\perp(i)}^{(I=0)} \right) V_{\perp(j)}^{\mu(I=0)} V_{\perp(k)}^{\nu(I=0)} \quad (\text{B.6})
\end{aligned}$$

where $a, b, c = 1, 2, 3$ corresponding to isospin, $i, j, k = 1, 2, 3$, and

$$V_\mu = V_{\mu\parallel} + V_{\mu\perp(1)} + V_{\mu\perp(2)} + V_{\mu\perp(3)} + V_{\mu\parallel}^{(I=0)} + V_{\mu\perp(1)}^{(I=0)} + V_{\mu\perp(2)}^{(I=0)} + V_{\mu\perp(3)}^{(I=0)}. \quad (\text{B.7})$$

Since the mixing structures of the gauge fields are given by

$$\begin{aligned}
V_{\mu\perp(3)}^a &= g(a_1)_\mu^a + \frac{r_{a_1}}{f_\pi} \partial_\mu \pi^a, \quad V_{\mu\parallel}^a = g \left(\cos \theta_\rho \rho_\mu^a + \sin \theta_\rho (\rho')_\mu^a \right), \quad V_{\mu\perp(1)}^a = g \left(\cos \theta_\rho (\rho')_\mu^a - \sin \theta_\rho \rho_\mu^a \right), \\
V_{\mu\perp(1)}^{(I=0)} &= g \left(\cos \theta_\omega (\omega')_\mu - \sin \theta_\omega \omega_\mu \right), \quad V_{\mu\perp(2)}^a = g(b_1)_\mu^a, \quad V_{\mu\perp(2)}^{(I=0)} = g(h_1)_\mu, \quad (\text{B.8})
\end{aligned}$$

the three-point interaction terms related to pion emission are expressed as

$$\begin{aligned}
\mathcal{L}_{\text{int}}^{(3)} &= -\frac{1}{2\sqrt{2}g} \epsilon^{abc} \left[\cos \theta_\rho \left(\partial_\mu \rho_\nu^a - \partial_\nu \rho_\mu^a \right) + \sin \theta_\rho \left(\partial_\mu (\rho')_\nu^a - \partial_\nu (\rho')_\mu^a \right) \right] \left(g(a_1)^{\mu b} + \frac{r_{a_1}}{f_\pi} \partial^\mu \pi^b \right) \left(g(a_1)^{\nu c} + \frac{r_{a_1}}{f_\pi} \partial^\nu \pi^c \right) \\
&\quad + \frac{1}{\sqrt{2}} \left[\cos \theta_\rho \left(\partial_\mu (\rho')_\nu^a - \partial_\nu (\rho')_\mu^a \right) - \sin \theta_\rho \left(\partial_\mu \rho_\nu^a - \partial_\nu \rho_\mu^a \right) \right] \left(g(a_1)^{\mu a} + \frac{r_{a_1}}{f_\pi} \partial^\mu \pi^a \right) (h_1)^\nu \\
&\quad - \frac{1}{\sqrt{2}} \left(\partial_\mu (b_1)_\nu^a - \partial_\nu (b_1)_\mu^a \right) \left(g(a_1)^{\mu a} + \frac{r_{a_1}}{f_\pi} \partial^\mu \pi^a \right) (\cos \theta_\omega (\omega')^\nu - \sin \theta_\omega \omega^\nu) \\
&\quad - \frac{1}{\sqrt{2}} \epsilon^{abc} \left(\partial_\mu (a_1)_\nu^a - \partial_\nu (a_1)_\mu^a \right) \left(g(a_1)^{\mu b} + \frac{r_{a_1}}{f_\pi} \partial^\mu \pi^b \right) (\cos \theta_\rho \rho^{\nu c} + \sin \theta_\rho (\rho')^{\nu c}) \\
&\quad - \frac{1}{\sqrt{2}} \left(\partial_\mu (h_1)_\nu - \partial_\nu (h_1)_\mu \right) \left(g(a_1)^{\mu a} + \frac{r_{a_1}}{f_\pi} \partial^\mu \pi^a \right) (\cos \theta_\rho (\rho')^{\nu a} - \sin \theta_\rho \rho^{\nu a}) \\
&\quad + \frac{1}{\sqrt{2}} \left[\cos \theta_\omega \left(\partial_\mu (\omega')_\nu - \partial_\nu (\omega')_\mu \right) - \sin \theta_\omega \left(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu \right) \right] \left(g(a_1)^{\mu a} + \frac{r_{a_1}}{f_\pi} \partial^\mu \pi^a \right) (b_1)^{\nu a} + \dots \quad (\text{B.9})
\end{aligned}$$

These interactions are controlled by only four parameters

$$g, \quad r_{a_1}, \quad \theta_\rho, \quad \theta_\omega, \quad (\text{B.10})$$

thanks to the SU(4) symmetry.

Appendix C: Intrinsic parity odd terms

$$\left(\hat{\alpha}_{\mu\perp(2)}^{(m)}, \hat{\alpha}_{\mu\perp(3)}, \hat{\alpha}_{\mu\perp(3)}^{(m)} \right) \rightarrow - \left(\hat{\alpha}_{\mu\perp(2)}^{(m)}, \hat{\alpha}_{\mu\perp(3)}, \hat{\alpha}_{\mu\perp(3)}^{(m)} \right). \quad (\text{C.1})$$

Intrinsic parity (IP) is a Z_2 transformation defined as

$$\left(\hat{\alpha}_{\mu\parallel}, \hat{\alpha}_{\mu\parallel}^{(m)}, \hat{\alpha}_{\mu\perp(1)}^{(m)} \right) \rightarrow + \left(\hat{\alpha}_{\mu\parallel}, \hat{\alpha}_{\mu\parallel}^{(m)}, \hat{\alpha}_{\mu\perp(1)}^{(m)} \right),$$

a_1 , and b_1 , which are the iso-triplet. The symbol “...” in Eq. (C.2) expresses that operators including the iso-singlet variables such as $\hat{\alpha}_{\mu\parallel}^{(I=0)}$ are also allowed. For convenience, we used the field strength $(V_{\mu\nu}^{(m)})$ defined as

$$\begin{aligned}
(V_{\mu\nu}^{(m)})_{\parallel} &\equiv 2\text{Tr} [\Xi_m \cdot V_{\mu\nu} \cdot \Xi_m^\dagger \cdot S^a] S^a, \\
(V_{\mu\nu}^{(m)})_{\perp(1)} &\equiv 2\text{Tr} [\Xi_m \cdot V_{\mu\nu} \cdot \Xi_m^\dagger \cdot X_{(1)}^a] X_{(1)}^a, \\
(V_{\mu\nu}^{(m)})_{\perp(2)} &\equiv 2\text{Tr} [\Xi_m \cdot V_{\mu\nu} \cdot \Xi_m^\dagger \cdot X_{(2)}^a] X_{(2)}^a, \\
(V_{\mu\nu}^{(m)})_{\perp(3)} &\equiv 2\text{Tr} [\Xi_m \cdot V_{\mu\nu} \cdot \Xi_m^\dagger \cdot X_{(3)}^a] X_{(3)}^a, \\
(V_{\mu\nu}^{(m)(I=0)})_{\parallel} &\equiv 2\text{Tr} [\Xi_m \cdot V_{\mu\nu} \cdot \Xi_m^\dagger \cdot S^0] S^0, \\
(V_{\mu\nu}^{(m)(I=0)})_{\perp(1)} &\equiv 2\text{Tr} [\Xi_m \cdot V_{\mu\nu} \cdot \Xi_m^\dagger \cdot X_{(1)}^0] X_{(1)}^0, \\
(V_{\mu\nu}^{(m)(I=0)})_{\perp(2)} &\equiv 2\text{Tr} [\Xi_m \cdot V_{\mu\nu} \cdot \Xi_m^\dagger \cdot X_{(2)}^0] X_{(2)}^0, \\
(V_{\mu\nu}^{(m)(I=0)})_{\perp(3)} &\equiv 2\text{Tr} [\Xi_m \cdot V_{\mu\nu} \cdot \Xi_m^\dagger \cdot X_{(3)}^0] X_{(3)}^0,
\end{aligned} \tag{C.3}$$

We can list the operators including $\hat{\alpha}_{\mu\parallel}^{(m)}$, $\hat{\alpha}_{\mu\parallel}^{(m)(I=0)}$, or external gauge field $\mathcal{V}_{\mu\nu}$ in a similar way.

Appendix D: Decays to $\pi + \gamma$

In this appendix, we calculate the decay widths of the spin-1 mesons to π and γ .

First we calculate the decay width of the $a_1 \rightarrow \pi\gamma$. The relevant $a_1\pi\gamma$, $a_1\rho\pi$ and $a_1\rho'\pi$ vertex functions are written as

$$\begin{aligned}
\Gamma^{(\mu a)(\nu)(c)}(a_1, \gamma, \pi) &= -e g_{a_1\mathcal{V}\pi} \epsilon^{a3c} g^{\mu\nu}, \\
\Gamma^{(\mu a)(\nu b)(c)}((p_{a_1}), (p_\rho), p_\pi) \\
&= g_{\rho a_1\pi} \epsilon^{abc} \left[(p_{a_1})^2 P^{\mu\nu}(p_{a_1}) - (p_\rho)^2 P^{\mu\nu}(p_\rho) \right], \\
\Gamma^{(\mu a)(\nu b)(c)}((p_{a_1}), (p_{\rho'}), p_\pi) \\
&= g_{\rho' a_1\pi} \epsilon^{abc} \left[(p_{a_1})^2 P^{\mu\nu}(p_{a_1}) - (p_{\rho'})^2 P^{\mu\nu}(p_{\rho'}) \right] \tag{D.1}
\end{aligned}$$

where $g_{a_1\mathcal{V}\pi} \equiv \frac{m_{a_1}^2 r_{a_1}}{g f_\pi}$. By using these effective vertex functions, we can calculate the amplitude of $a_1 \rightarrow \pi\gamma$;

$$\begin{aligned}
\mathcal{M}(a_1(p_{a_1})_\mu^a \rightarrow \pi(p_\pi)^b \gamma(p_\gamma)_\nu) \\
= e \epsilon^{ab3} \epsilon_\nu(p_\gamma)^* \epsilon_\mu(p_{a_1}) \\
\times g^{\mu\nu} \left(-g_{a_1\mathcal{V}\pi} + \frac{g_\rho}{m_\rho^2} g_{\rho a_1\pi} + \frac{g_{\rho'}}{m_{\rho'}^2} g_{\rho' a_1\pi} \right) \tag{D.2}
\end{aligned}$$

where $\epsilon^\mu(p)$ is the polarization vector and we used $p_\gamma^2 = 0$. Then, by using Eqs. (IV.9) and (VI.6), the above amplitude vanishes: $\mathcal{M}(a_1(p_{a_1})_\mu^a \rightarrow \pi(p_\pi)^b \gamma(p_\gamma)_\nu) = 0$. This implies that the decay width for $a_1 \rightarrow \pi\gamma$ vanishes;

$$\Gamma(a_1 \rightarrow \pi\gamma) = \frac{|\vec{p}_\pi|}{8\pi m_{a_1}^2} \frac{1}{9} \sum |\mathcal{M}|^2 = 0. \tag{D.3}$$

The model with the generalized HLS [10] also shows that the partial width of $a_1 \rightarrow \pi\gamma$ vanishes at the $\mathcal{O}(p^2)$ order.

Since the mass of the initial particle is around 1GeV while $m_\pi \simeq 137\text{MeV}$ and $m_\gamma = 0$, the momentum of the outgoing particles is about 1GeV. This means that the higher order contribution of the derivative expansion may not be small. The $\mathcal{O}(p^4)$ contribution is estimated as $\mathcal{M}^{\mathcal{O}(p^4)} \sim 100\text{MeV}$ because of $\left(\frac{|\vec{p}_\pi|}{4\pi f_\pi}\right)^2 \simeq \left(\frac{m_{a_1}/2}{4\pi f_\pi}\right)^2 \simeq 0.3$ and $\left|\frac{em_{a_1}^2 r_{a_1}}{g f_\pi}\right| \sim 330\text{MeV}$. One can find that $\Gamma(a_1 \rightarrow \pi\gamma)$ is of order 100keV. Therefore, the deviation between our result and the experiments, $\Gamma(a_1 \rightarrow \pi\gamma) = 640 \pm 246\text{keV}$, is understood as the contribution of the $\mathcal{O}(p^4)$ order.

Similarly to the $a_1 \rightarrow \pi\gamma$ decay, the $b_1 \rightarrow \pi\gamma$ and $h_1 \rightarrow \pi\gamma$ amplitudes vanish at the $\mathcal{O}(p^2)$ order, respectively;

$$\begin{aligned}
\mathcal{M}((b_1)^a \rightarrow \pi^b \gamma) \\
= e \delta^{ab} \epsilon_\nu(p_\gamma)^* \epsilon_\mu(p_{b_1}) g^{\mu\nu} \left(\frac{g_\omega}{m_\omega^2} g_{\omega b_1\pi} + \frac{g_{\omega'}}{m_{\omega'}^2} g_{\omega' b_1\pi} \right) = 0, \\
\mathcal{M}(h_1 \rightarrow \pi^0 \gamma) \\
= e \epsilon_\nu(p_\gamma)^* \epsilon_\mu(p_{h_1}) g^{\mu\nu} \left(\frac{g_\rho}{m_\rho^2} g_{\rho h_1\pi} + \frac{g_{\rho'}}{m_{\rho'}^2} g_{\rho' h_1\pi} \right) = 0 \tag{D.4}
\end{aligned}$$

where we used $p_\gamma^2 = 0$ and Eqs. (IV.9), (VI.6), and (VI.7). Then, their decay widths are

$$\Gamma(b_1 \rightarrow \pi\gamma) = \Gamma(h_1 \rightarrow \pi\gamma) = 0. \tag{D.5}$$

The empirical value $\Gamma(b_1 \rightarrow \pi\gamma) = 230 \pm 60\text{keV}$ is the same order as the $\mathcal{O}(p^4)$ contribution estimated for the decay of the a_1 meson.

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